17 July 2000

PHYSICS LETTERS A

Physics Letters A 272 (2000) 39–45

www.elsevier.nl/locate/pla

Damping dilution factor for a pendulum in an interferometric gravitational waves detector

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Received 27 April 2000; accepted 12 June 2000
Communicated by P.R. Holland

Abstract

Mechanical loss in pendulums is a subject of great importance to gravitational waves detectors being built and being planned, as this determines the level of thermal noise associated with the detector suspensions. Relationships between the mechanical loss of the pendulum and the mechanical loss of the suspending fibres or wires can be derived in two apparently contradictory ways which give answers different by a factor of two. In this paper the differences are resolved and it is shown that both methods lead to the same answer.

1. Introduction

It is well known that the loss factor of a pendulum is considerably lower than the loss factor of the wires used to suspend it. This is a result of most of the energy being stored in lossless gravitational potential energy with only a small part of the energy being stored in the bending of the suspending wires. Relationships between the loss of the pendulum and the loss of the wires have been derived by a number of researchers, see for example Saulson [1], Braginsky [2] and Gonzalez and Saulson [3]. However there are aspects of the derivation of these relationships which require further discussion of the basic physics, as it is possible to approach the problem in two different ways and obtain relationships which apparently differ by a factor of two. One method of calculating the loss of the pendulum from the loss of the material of the wire involves using the ratio between the potential energy stored in the wire to the total potential energy stored in the system; the other method involves expressing the effective spring constants of the pendulum and the wire as complex quantities with the relevant loss factor being expressed as the ratio between the imaginary to the real part. In this paper we shall address this issue and show that there is a misconception in estimating the overall spring constant of the pendulum in too approximate a manner.

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0375-9601/00/$ - see front matter
PII: S0375-9601(00)00411-4

2. Highlighting the problem

In the equation of motion of a pendulum it is possible to take damping into account by introducing an imaginary restoring force [4].

The ratio between the imaginary and the real restoring force (thus, spring constant) at any frequency is called the *Loss Angle or Factor* ϕ and it is usually chosen as a parameter suitable to quantify the level of dissipation in a particular dynamical system. In a simple harmonic oscillator, like a pendulum, the inverse of the loss angle is the quality factor Q of the resonance where the quality factor is proportional to the ratio of the energy stored in a system to the energy dissipated per unit time.

To be more detailed, the same argument can be applied to the elastic properties of a material; in this case the Young's modulus Y is considered a complex number rather than real and the ratio of the imaginary to the real part of Y, namely the loss angle, is a characteristic parameter of the material. Typical values are between 10^{-6} and 10^{-3}. Our discussion is centered on the relation between the pendulum loss angle ϕ_{pend} and the loss angle ϕ_{mat} of the material of the suspension wire.

The problem being discussed is most easily highlighted by following Saulson [1] for the case of a simple pendulum consisting of a mass joined well above its centre of mass to a single suspension wire. It should be noted that for such a system the bending of the wire as the pendulum swings is localised close to the top of the wire.

In order to work out the loss angle of the pendulum, one has to solve the elastic Eq. (A.1) of a thin beam longitudinally pre-stressed. Detailed calculations are shown in Appendix A where we obtained the horizontal wire displacement y from the vertical position as a function of the time t and the vertical position along the x axis (see Fig. 1):

$$y(x,t) = y(x)e^{i\omega t} = \frac{F}{T\lambda}\left[e^{-\lambda x} + \lambda x - 1\right]e^{i\omega t} \quad (1)$$

F is the amplitude of the sinusoidal force $F\exp(i\omega t)$ applied horizontally at the end of the wire, T is the tension in the wire, $\lambda = \sqrt{T/Y_0 I}$, Y_0 is the magnitude of the Young's modulus of the material and I is the moment of the cross section of the wire. This equation is valid for low frequencies where the inertia of the wire can be considered negligible.

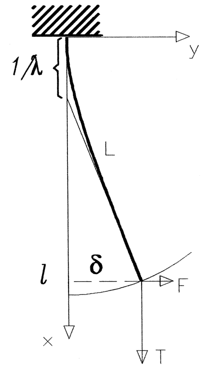

Fig. 1. Schematic of the deformation of the wire once it is pulled apart by the force F. The tension T is the weight of the suspended pointlike mass. The pendulum swings with an effective length smaller than the real length L by the amount $1/\lambda$.

Once the deformation of the wire is known, the stored elastic energy V_{el} is worked out using the relation [5]:

$$V_{\text{el}} = \frac{1}{2} Y_0 I \int_0^L \left(\frac{d^2 y(x)}{dx^2} \right)^2 dx \simeq \frac{1}{2}\left(\frac{T}{2L}\sqrt{\frac{Y_0 I}{TL^2}} \right)\delta(t)^2 \quad (2)$$

where L is the length of the wire and δ is the horizontal displacement at its end. Through the action of the tension T given by the weight of the suspended mass, energy is also stored in the gravitational field. This energy is:

$$V_g = \frac{1}{2}\frac{T}{L}\delta(t)^2 \quad (3)$$

In our case, for the same displacement δ, the energy stored in the gravitational field is much more than the energy stored in the elastic deformation of the wire. Therefore the total potential energy V_t can be aproximated by V_g. As discussed in Appendix B the ratio between the elastic energy, that is partially dissipated, and the total potential energy, averaged over one period of oscillation, as indicated by the bars over the symbols, gives the scaling factor from the material loss angle to the pendulum one.

$$\phi_{\mathrm{pend}} = \phi_{\mathrm{mat}} \frac{\overline{V_{\mathrm{el}}}}{\overline{V_{\mathrm{t}}}} \simeq \phi_{\mathrm{mat}} \frac{\overline{V_{\mathrm{el}}}}{\overline{V_{\mathrm{g}}}} = \frac{\phi_{\mathrm{mat}}}{2} \sqrt{\frac{Y_0 I}{TL^2}} \tag{4}$$

The scaling factor is also called *dilution factor*. In the equation above it is $2\sqrt{TL^2/Y_0 I}$.

Both V_{el} (2) and V_{g} (3) depend quadratically on the displacement δ. Therefore from the expressions above one can define a spring constant for each of the energies since the general relation $V = k\delta^2/2$ is assumed. The two spring constants are:

$$k_{\mathrm{el}} = \frac{T}{2L} \sqrt{\frac{Y_0 I}{TL^2}} \tag{5}$$

$$k_{\mathrm{g}} = \frac{T}{L} \tag{6}$$

In our case, it easy to see that the ratio between potential energies is equal to the ratio between the respective spring constants and then the Eq. (4) can be replaced by $\phi_{\mathrm{pend}} = \phi_{\mathrm{mat}} k_{\mathrm{el}}/k_{\mathrm{g}}$ obtaining the same result.

Another way to work out the pendulum loss angle is to consider the fact that in the frequency domain the Young's modulus is complex, $Y = Y_0(1 + i\phi_{\mathrm{mat}})$. For $\phi_{\mathrm{mat}} \ll 1$ the beam equation (see Eq. (A.1) in Appendix A) is still assumed valid [6] provided Y_0 is replaced by Y.

Doing this the spring constants (and the energies) are complex. The imaginary part of the spring constant represents the dephasing between the displacement of the spring and the force applied at, due to the anelastic behaviour of the material. Any dephasing between force and displacement is equivalent to an energy dissipation.

Using the aproximation $\sqrt{Y_0(1 + i\phi_{\mathrm{mat}})} \simeq \sqrt{Y_0}(1 + i\phi_{\mathrm{mat}}/2)$ the total spring constant $k_{\mathrm{pend}} = k_{\mathrm{g}} + k_{\mathrm{el}}$ is:

$$\begin{aligned} k_{\mathrm{pend}} &= \frac{T}{L}\left(1 + \frac{1}{2}\sqrt{\frac{YI}{TL^2}}\right) \\ &\simeq \frac{T}{L}\left[1 + \tfrac{1}{2}\sqrt{\frac{Y_0 I}{TL^2}}\left(1 + i\,\frac{\phi_{\mathrm{mat}}}{2}\right)\right] \end{aligned} \tag{7}$$

In our case the term $\sqrt{Y_0 I/TL^2} \ll 1$ and then k_{pend} can be written:

$$k_{\mathrm{pend}} \simeq \frac{T}{L}\left(1 + i\,\frac{\phi_{\mathrm{mat}}}{4}\sqrt{\frac{Y_0 I}{TL^2}}\right) \tag{8}$$

Then from the above equation the pendulum loss angle is the phase angle of the total (complex) spring constant, from the previous equation:

$$\phi_{\mathrm{pend}} = \frac{\phi_{\mathrm{mat}}}{4}\sqrt{\frac{Y_0 I}{TL^2}} \tag{9}$$

A factor of 2 difference from the expression in Eq. (4) is clearly seen. Interestingly, this result agrees with that from a derivation based on complex energy expressions. Using the same aproximation for Young's modulus as above one can write the elastic energy (2) as:

$$\begin{aligned} V_{\mathrm{el}} &\simeq \frac{1}{2}\left[\frac{T}{2L}\sqrt{\frac{Y_0 I}{TL^2}}\left(1 + i\,\frac{\phi_{\mathrm{mat}}}{2}\right)\right]\delta^2 \\ &= V_{\mathrm{el}}^0\left(1 + i\,\frac{\phi_{\mathrm{mat}}}{2}\right) \end{aligned} \tag{10}$$

Where V_{el}^0 is the real part of the elastic energy. From the considerations above, just the imaginary part $\phi_{\mathrm{mat}} V_{\mathrm{el}}^0/2$ represents the energy dissipated and then one can write Eq. (4) as:

$$\phi_{\mathrm{pend}} \simeq \phi_{\mathrm{mat}} \frac{\overline{V_{\mathrm{el}}^0}/2}{\overline{V_{\mathrm{g}}}} = \frac{\phi_{\mathrm{mat}}}{4}\sqrt{\frac{Y_0 I}{TL^2}} \tag{11}$$

which is the same as the Eq. (9)

3. The solution

In order to find out where the discrepancy is, it is better to check carefully the validity of the relations so far used.

Firstly, in Appendix B it is shown that for an anelastic body having a material loss angle ϕ_{mat} and undergoing a periodic stress, the cycle-averaged dissipated energy is proportional to the cycle-averaged stored elastic energy through the constant $4\pi\phi_{\mathrm{mat}}$ (Eq. (B.13)).

Therefore, equating the dissipated energy written either as proportional to the elastic energy (B.13) or proportional to the total potential energy through the constant $4\pi\phi_{\rm pend}$,

$$4\pi\,\phi_{\rm mat}\,\overline{V_{\rm el}} = 4\pi\,\phi_{\rm pend}\,\overline{V_{\rm t}} \tag{12}$$

and, the energy relation (4) is easily obtained.

Secondly, from the solution (1) of the beam equation the total spring constant of the pendulum $k_{\rm Pend}$ can be worked out since $k_{\rm Pend}$ is defined as F/δ where δ is the displacement of the free end of the wire. The result is (see Appendix A):

$$k_{\rm Pend} = \frac{T}{L}\left(1 + \sqrt{\frac{YI}{TL^2}}\right) \tag{13}$$

As can be seen, the previous result is different from the first equation in (7) by a factor of 2 outside the square root term.

Applying the same aproximation for $\sqrt{Y}$ used earlier to (13) a pendulum loss angle $\phi_{\rm pend}$ equal to the result (4) is found.

At this point it seems that the discrepancy is due to an incorrect expression (7) used previously for $k_{\rm Pend}$. In fact, detailed calculations reported in Appendix A show that the rigidity of the wire produces a shortening effect on the effective length of the pendulum. As a consequence, new expressions for the gravitational energy $V_{\rm G}$ and the gravitational spring constant $k_{\rm G}$ have to be introduced:

$$V_{\rm G} = \frac{1}{2}\,\frac{T}{L}\left(1 + \frac{1}{2}\sqrt{\frac{YI}{TL^2}}\right)\delta^2 \tag{14}$$

$$k_{\rm G} = \frac{T}{L}\left(1 + \frac{1}{2}\sqrt{\frac{YI}{TL^2}}\right) \tag{15}$$

In this new expression for the gravitational energy, the complex Young's modulus of the material is present and this means that some energy is dissipated. The same happens to the elastic energy $V_{\rm el}$ (10).

In order to clarify the relation between the complex phase of the spring constant and the energy dissipation rate, firstly it can be seen that both $k_{\rm G}$ and $k_{\rm el}$ have expressions of the form $k = k_0\,(1 + i\phi)$ where k_0 is real and positive. Then the restoring force acting on the mass is:

$$F_{\rm R}(t) = \Re\left\{-k_0\,(1 + i\,\phi)\;\delta(t)\right\} \tag{16}$$

where the displacement δ is:

$$\delta(t) = \Re\left\{a\,{\rm e}^{i\omega t}\right\} \tag{17}$$

a being the complex amplitude of oscillation. The instantaneous power provided by the force is $P(t) = F_{\rm R}(t)\,\dot\delta(t)$ where $\dot\delta(t)$ is the velocity of the mass. Once the relations (16) and (17) have been used in the power expression and after an integration in time over one period $\tau = 2\pi/\omega$ of oscillation is done, then the energy $\overline{V_{\rm d}}$ provided by the force $F_{\rm R}$ is:

$$\overline{V_{\rm d}} = -2\pi\,\phi\,k_0\,\frac{aa^*}{2} \tag{18}$$

The negative sign signifies that energy is being lost. Both the gravitational and the elastic forces supply energy to the pendulum and once k_0 and ϕ are replaced by the relevant expressions, the energy losses can be added to give the total lost energy $\overline{V_{\rm d}^{\rm T}}$ per cycle:

$$\overline{V_{\rm d}^{\rm T}} = \overline{V_{\rm d}^{G}} + \overline{V_{\rm d}^{\rm el}} \simeq -2\pi\,\frac{T}{L}\sqrt{\frac{Y_0\,I}{TL^2}}\;\frac{\phi_{\rm mat}}{2}\;\frac{aa^*}{2} \tag{19}$$

for $\sqrt{Y_0\,I/TL^2} \ll 1$. With the same aproximation the total potential energy averaged in one cycle is $\overline{V_{\rm t}} \simeq T/L\cdot aa^*/4$. Then the Eq. (19) can be written in the following way:

$$\overline{V_{\rm d}^{\rm T}} = -4\pi\sqrt{\frac{Y_0\,I}{TL^2}}\;\frac{\phi_{\rm mat}}{2}\;\overline{V_{\rm t}} = -4\pi\,\phi_{\rm pend}\,\overline{V_{\rm t}} \tag{20}$$

from which the same loss factor of the pendulum as in (4) can be worked out.

This method of identifying the dissipation mechanism given in Appendix B gives the same result as the method of the spring constant dephasing.

4. Conclusion

If the spring constant for the pendulum is calculated in full there is no discrepancy between the loss of the pendulum calculated by the two different methods. For a pointlike mass the result is:

$$\phi_{\rm pend} = \frac{\phi_{\rm mat}}{2}\sqrt{\frac{Y_0\,I}{TL^2}} \tag{21}$$

It is interesting to note, however, that, although in this model all the dissipation mechanisms are inside the wire, the stiffness of the wire affects the energy stored both in the wire bending and in the vertical position of the mass, when the horizontal deflection of the lower end of the wire is chosen as a dynamic variable.

The method of identifying the dissipation mechanism gives the energy dissipated in a particular part of the system. In the case of the pendulum this part is the wire and it is the only part able to dissipate energy.

The other method uses the complex phase of the spring constant. If this method is applied to the total spring constant, it gives the overall energy lost by the forces acting on the system whereas, if it is applied to each spring constant, it gives the energy lost by the relevant force.

The sum of the energies lost by the external forces is equal to the energy dissipated.

Appendix A.

If a small force $F(t)$ is applied to the end of a thin rod with a linear mass density ρ, under a tension $T=mg$ as shown in Fig. 1, the deflection $y(x,t)$ of the rod is the solution of the following equation [5]:

$$YIy'''' - Ty'' = \rho \frac{\partial^2 y}{\partial t^2} \tag{A.1}$$

where each apostrophe stands for a derivative with respect to x. In the case of harmonic excitation $F(t) = F\,\mathrm{e}^{i\omega t}$ it is possible to find the solution as $y(x,t) = y(x)\mathrm{e}^{i\omega t}$. The boundary conditions are: $y(x=0)=0$ and $y'(x=0)=0$ for the upper clamp; $y''(x=L)=0$ (no torque) and $YIy'''(x=L) - Ty'(x=L) = F$ for the free end. For sufficiently small angular frequency ω the inertial term (i.e. the r.h.s) of the Eq. (A.1) may be considered negligible compared to the other terms. In this approximation and considering the boundary conditions, the deformation $y(x)$ is given by:

$$y(x) = \frac{F}{T\lambda}\left[\mathrm{e}^{-\lambda x} + \lambda x - 1\right] \tag{A.2}$$

where $\lambda = \sqrt{T/YI}$. In all the following calculations it is assumed that $\lambda L \gg 1$. The wire length L and its projection l on the vertical axis x are related as:

$$L = \int_0^l \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx \simeq \int_0^l \left[1+\frac{1}{2}\left(\frac{dy}{dx}\right)^2\right]dx$$
$$\simeq l\left[1+\frac{1}{2}\left(\frac{\delta}{l}\right)^2\left(1+\frac{1}{2\lambda l}\right)\right] \tag{A.3}$$

where $\delta = y(l)$. The inverse of the previous equation is:

$$l \simeq L\left[1-\frac{1}{2}\left(\frac{\delta}{L}\right)^2\left(1+\frac{1}{2\lambda L}\right)\right] \tag{A.4}$$

where all the terms of order greater than 2 in the power of δ/L are neglected.

From Eq. (A.2) it is possible to extract the equivalent spring constant of the pendulum k_{Pend}:

$$k_{\mathrm{Pend}} = \frac{F}{\delta} = \frac{T\lambda}{\mathrm{e}^{-\lambda l} + \lambda l - 1} \tag{A.5}$$

Considering that the replacement of l with L introduces a negligible error of order $\delta/L \ll 1$ and that $\lambda L \gg 1$, the previous equation can be simplified as

$$k_{\mathrm{Pend}} \simeq \frac{T}{L}\left(1+\frac{1}{\lambda L}\right) \tag{A.6}$$

The approximation $l \simeq L$ restricts validity to the linear regime.

Now consider the estimation of the gravitational and elastic energies when the wire is bent. Referring to the Fig. 1 and using the Eq. (A.4), the gravitational potential of the suspended mass is given by:

$$V_{\mathrm{g}} = mg(L-l) = \frac{1}{2}\frac{T}{L}\left(1+\frac{1}{2\lambda L}\right)\cdot\delta^2 \tag{A.7}$$

The elastic energy due to the bending is:

$$V_{\mathrm{el}} = \tfrac{1}{2}YI\int_0^l\left(\frac{d^2y}{dx^2}\right)^2 dx \simeq \frac{1}{2}\frac{T}{L}\left(\frac{1}{2\lambda L}\right)\delta^2 \tag{A.8}$$

where the last approximation is valid when $\lambda L \gg 1$ and $\delta/L \ll 1$. Then, the total potential energy V_{t} is the sum of the gravitational and bending energies:

$$V_{\mathrm{t}} = \frac{1}{2}\frac{T}{L}\left(1+\frac{1}{\lambda L}\right)\cdot\delta^2 \tag{A.9}$$

It is worthwhile to note the quadratic dependence of these energies on the displacements δ; then a spring constant for each of these energies can be extracted using the general relation $V = \frac{1}{2} k x^2$ valid for the harmonic oscillator. In the case of the total energy (A.9) this yields a pendulum spring constant the same as that Eq. (A.6).

For the gravitational and elastic energies, the resulting spring constants k_G and k_{el} read:

$$k_G = \frac{T}{L}\left(1 + \frac{1}{2\lambda L}\right) \tag{A.10}$$

$$k_{el} = \frac{T}{L}\left(\frac{1}{2\lambda L}\right) \tag{A.11}$$

It is easy to see that adding k_G to k_{el}, the expression of k_{Pend} given in the Eq. (A.6) is obtained.

Appendix B. General formulation of energy dissipation in an anelastic body

In any deformed elastic body the rate at which the work is done in an infinitesimal element is p:

$$p = \sigma_{ij} \frac{\partial \epsilon_{ij}}{\partial t} \tag{B.1}$$

where σ is the stress and ϵ is the strain, both of them being temporally and spatially dependent. For a sinusoidal motion at angular frequency ω one can write the stress as $\sigma_{ij}(t) = \sigma_{ij} \exp i \omega t$ and the strain as $\epsilon_{ij}(t) = \epsilon_{ij} \exp i \omega t$ where ϵ and σ are now the amplitudes of the respective quantities. The cycle average of p will be the dissipated power p_d:

$$p_d = \frac{\omega}{2} \Im \left\{ \sigma_{ij}\, \epsilon_{ij}^* \right\} \tag{B.2}$$

In general the stress and strain amplitudes are complex since it is necessary to take into account the phase. In the same way, the cycle average of the stored energy density e_s is[1]:

$$e_s = \tfrac{1}{4} \Re \left\{ \sigma_{ij}\, \epsilon_{ij}^* \right\} \tag{B.3}$$

[1] Valid for a linear material.

The general constitutive relation of any linear material is given by

$$\sigma_{ij}(t) = \int_{-\infty}^{t} c_{ijkl}(t - t')\, \epsilon_{kl}(t')\, dt' \tag{B.4}$$

where the kernel of the convolution integral $c_{ijkl}(z)$ is the time behaviour of the stress when a δ function strain is applied. Considering causality, $c_{ijkl}(z)$ is always null for $z < 0$ and for a perfect elastic material it reduces to the simple form $c_{ijkl}\, \delta(z)$. For a homogeneous material the stiffness tensor c_{ijkl} is [7]:

$$c_{ijkl} = \mu\left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk} \right) + \lambda\, \delta_{ij}\delta_{kl} \tag{B.5}$$

where μ and λ are the Lame's constants.

In the case of a sinusoidal oscillation $\epsilon_{kl} \exp(i\,\omega s)$, Eq. (B.4) is:

$$\begin{aligned} \sigma_{ij} &= \tilde{C}_{ijkl}(\omega)\, \epsilon_{kl}\, e^{i\omega t} \\ &= \left[C_{ijkl}(\omega) + i C_{ijkl}(\omega)' \right] \epsilon_{kl}\, e^{i\omega t} \end{aligned} \tag{B.6}$$

where $\tilde{C}_{ijkl}(\omega)$ is the Fourier transform of $c_{ijkl}(z)$, written with its real and imaginary parts in the second row. Once Eq. (B.6) is inserted into Eq. (B.2) and Eq. (B.3), we find, respectively, for the cycle-averaged dissipated power density and for the stored energy:

$$p_d = \frac{\omega}{2} \Im \left\{ \tilde{C}_{ijkl}(\omega)\, \epsilon_{kl} \epsilon_{ij}^* \right\} \tag{B.7}$$

$$e_s = \tfrac{1}{2} \Re \left\{ \tilde{C}_{ijkl}(\omega)\, \epsilon_{kl} \epsilon_{ij}^* \right\} \tag{B.8}$$

In order to calculate the imaginary and real parts of the last equations it is worthwhile to use the following mathematical relation:

$$\begin{aligned} &\tilde{C}_{ijkl}(\omega)\, \epsilon_{kl} \epsilon_{ij}^* \\ &\quad = C_{ijkl}(\omega) \Re \left\{ \epsilon_{kl} \epsilon_{ij}^* \right\} + i\, C'_{ijkl}(\omega) \Re \left\{ \epsilon_{kl} \epsilon_{ij}^* \right\} \end{aligned} \tag{B.9}$$

based on the symmetry $\tilde{C}_{ijkl} = \tilde{C}_{klij}$ [7].

The integral over the volume of the Eq. (B.7) and Eq. (B.8) gives respectively the total average power dissipated $\overline{P_d}$ and the total average elastic energy $\overline{V_{el}}$.

Noting that the period of the motion τ is $2\pi/\omega$ and using the Eq. (B.9), the total energy dissipated in a cycle is:

$$P_{\mathrm{d}}\,\tau = \pi\, C'_{ijkl}(\omega) \int \Re\left\{\epsilon_{kl}\epsilon^{*}_{ij}\right\} d\mathscr{V} \tag{B.10}$$

$$= 4\pi\, C'_{ijkl}(\omega)\, C^{-1}_{ijkl}(\omega)\, \overline{V_{\mathrm{el}}} \tag{B.11}$$

where the second form follows from the first by comparison with Eq. (B.8) and from the definition of $\overline{V_{\mathrm{el}}}$. In the special case where for each of the components of the stiffness tensor the real and the imaginary parts are in the same ratio, i.e. if

$$C'_{ijkl}(\omega) = \phi_{\mathrm{mat}}(\omega)\, C_{ijkl}(\omega) \tag{B.12}$$

Eq. (B.11) can be rewritten as

$$P_{\mathrm{d}}\,\tau = 4\pi\, \phi_{\mathrm{mat}}(\omega)\, \overline{V_{\mathrm{el}}} \tag{B.13}$$

and the quality factor Q takes the usual form,

$$Q = 2\pi \frac{\overline{V_{\mathrm{el}}} + \overline{K}}{P_{\mathrm{d}}\,\tau} = 2\pi \frac{2\,\overline{V_{\mathrm{el}}}}{P_{\mathrm{d}}\,\tau} = \frac{1}{\phi_{\mathrm{mat}}(\omega)} \tag{B.14}$$

where $\overline{K}$ is the average kinetic energy which is equal to $\overline{V_{\mathrm{el}}}$ for a free harmonic oscillator. Since this result has been obtained from a generic condition, it means that, whatever strain distribution is considered, the ratio of the dissipated power to the potential energy stored, both inside the body, leads to the material loss angle ϕ_{mat}. When the body is under another conservative force, then the amount of energy stored increases and a dilution factor appears. In the case of the pendulum, the other conservative energy is the gravitational one V_{g} and then the pendulum loss angle ϕ_{pend} is:

$$\phi_{\mathrm{pend}} = \frac{P_{\mathrm{d}}\,\tau}{4\pi \overline{V_{\mathrm{t}}}} = \frac{P_{\mathrm{d}}\,\tau}{4\pi\left(\overline{V_{\mathrm{el}}} + \overline{V_{\mathrm{g}}}\right)} = \phi_{\mathrm{mat}} \frac{\overline{V_{\mathrm{el}}}}{\overline{V_{\mathrm{el}}} + \overline{V_{\mathrm{g}}}} \tag{B.15}$$

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