Thermoelastic dissipation in inhomogeneous media: loss measurements and displacement noise in coated test masses for interferometric gravitational wave detectors

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The displacement noise in the test-mass mirrors of interferometric gravitational wave detectors is proportional to their elastic dissipation at the observation frequencies. In this paper, we analyze one fundamental source of dissipation in thin coatings, thermoelastic damping associated with the dissimilar thermal and elastic properties of the film and the substrate. We obtain expressions for the thermoelastic dissipation factor necessary to interpret resonant loss measurements, and for the spectral density of displacement noise imposed on a Gaussian beam reflected from the face of a coated mass. The predicted size of these effects is large enough to affect the interpretation of loss measurements, and to influence design choices in advanced gravitational wave detectors.

I. INTRODUCTION

While recent results indicate that the elastic losses available in bulk materials such as silica and sapphire are adequate to achieve the design goals for displacement noise in next generation interferometric gravitational wave detectors, the losses associated with the multilayer dielectric mirrors deposited on the faces of the mass are large enough to contribute significantly to the total noise of the system. The origin of these coating losses is not yet clear. In this paper, we investigate an intrinsic dissipation mechanism, thermoelastic effects associated with a thin film on a bulk substrate. Thermoelastic dissipation has been known since Zener's work in the 1930s [1]. In homogeneous solids, it is associated with the irreversible flow of heat driven by temperature gradients associated with strain gradients in the solid. These effects lead to the well-known result for the damping of flexural vibrations in a thin beam, where the heat flows from the side of the beam in compression to the side in tension [2]. The maximum dissipation $\phi_{\max}$ in this case is a function only of the material properties and not the beam dimensions, $\phi_{\max} = Q^{-1} \sim E\alpha^2 T/C$, where E is the Young's modulus, α is the coefficient of linear thermal expansion, C is the volumetric heat capacity, and T is the background temperature. The dissipation peak occurs at a frequency $\omega_p \sim 1/\tau$ where $\tau \sim l^2 C/k$ is the thermal diffusion time through a beam of thickness l given the thermal conductivity k.

More recently, thermoelastic dissipation in homogeneous test masses has been analyzed as a source of noise in gravitational wave detectors [3], where the heat can be viewed as diffusing radially in the compressed region associated with a Gaussian-distributed pressure field on the surface of the mass, as is used in Levin's method for analyzing displacement noise [4]. Characteristic of both these examples is the presence of a nonuniform strain field, necessary to create a temperature gradient to drive the thermal diffusion in a homogeneous medium.

In an inhomogeneous body, temperature gradients can be generated in a *uniform* strain field, so that thermoelastic dissipation can be expected even in the absence of stress or strain gradients. For the case of a coated test mass, if the thermoelastic properties of the film are different from those in the substrate, we can anticipate that thermal diffusion and hence thermoelastic dissipation will occur. An estimate of the size of the effect can be obtained by comparison with the flexural damping of a thin beam. Replacing the thermal expansion coefficient by the difference between these coefficients in the film and substrate, and assuming for simplicity that the other pertinent material parameters are the same, we have $\phi_{\max} \sim E(\alpha_f - \alpha_s)^2 T/C$, and again expect the peak response to occur for frequencies corresponding to the thermal diffusion time through the film. If we consider a film with the thermal expansion coefficient of alumina on a substrate with the thermal expansion of silica, and take the other parameters to be those of silica we find $\phi_{\max} \sim 3 \times 10^{-4}$, comparable to the elastic losses measured in optical coatings [5,6]. For a 5-micron-thick film with these properties the dissipation maximum occurs at ~ 5 kHz, corresponding to the thermal diffusion time through the film. This frequency is in the range typically sampled by mechanical loss measurements, and not far from the frequency band of interest for gravitational wave detection.

It appears that a more quantitative investigation of these effects is necessary to evaluate their implications for characterization of test masses as well as for gravitational wave detectors themselves. We consider in this paper two questions associated with the thermoelastic mechanism: what is the effective dissipation in the situation characteristic of resonator measurements of elastic loss, and what is the expected spectral density of displacement noise in the situation characteristic of a test mass in a gravitational wave detector.

The key results from the analyses contained here are: (a) the derivation of an expression for $\phi_{tot,\parallel}$, the thermoelastic dissipation expected in a coated test mass undergoing deformations of the type expected in mechanical loss measurements and (b) the derivation of an expression for $S_x(f)$, the power spectral density of thermoelastic displacement noise associated with the dielectric mirror coating on a test-mass substrate.

A recent independent calculation of the spectral density of thermal noise in the low-frequency limit agrees, for the case where the thermoelastic properties of the film and substrate other than the thermal expansion coefficient are identical, with the results given here taken to that same limit. [7,8].

In Ref. [8], the difference between the expressions for thermoelastic thermal noise presented here and in [7] for cases where the elastic properties of the film and substrate differ is noted. After discussions with the authors this difference has been resolved in favor of the results presented here.

II. SKETCH OF THE CALCULATION

In this section, we sketch the calculation of the thermoelastic dissipation and the displacement noise in an inhomogeneous medium. Details of the calculation are given in Sec. IV and Appendices.

There are three steps to calculating the thermoelastic loss in the coating:

(1) Obtain the oscillatory thermal field associated with the zeroth-order elastic fields,
(2) Calculate the complex first-order elastic fields generated by the spatially varying oscillatory thermal field, and then
(3) Calculate the power dissipated by the interaction of the zeroth- and first-order elastic fields.

Throughout we will consider only linear thermoelasticity, retaining terms up to first-order in the oscillatory thermal field. Therefore the stored energy can be taken to be proportional to the square of zeroth-order elastic fields, while the imaginary part of the product of the zeroth-order elastic fields and the elastic fields induced by the thermal wave represent the relevant average dissipated power. Zeroth and first-order quantities are denoted by subscripts 0, 1, respectively.

The geometry we consider consists of a film of thickness l and coefficient of linear thermal expansion α_f on a substrate with α_s whose thermophysical properties are all possibly different from those of the film. We take the surface normal to be in the $-z$ direction, and the surface located at $z = 0$, so that the film extends from $z = 0$ to $z = l$, and the substrate from $z = l$ to $z = \infty$. This is shown in Fig. 1.

To simplify the analysis, we assume that the multilayer film can be approximated as a uniform film with appropriately averaged properties, and assume that the film is thin enough and the thermal diffusion length at the frequencies of interest short enough compared to any relevant transverse dimension (e.g. the dimensions of the object itself, or the radius of the Gaussian beam interrogating its surface) that only the thermal diffusion normal to the surface of the mass need be considered. Since the thermal diffusion lengths for frequencies of interest are on the order of or longer than the total film thickness, the description of the film in terms of its average properties appears reasonable, but we also consider the case of a film whose thermal expansion (but no other parameters) varies periodically through its thickness, as a simple model to explore any unexpected effects that might arise from the neglected microstructure of the film.

The point of departure for the calculation is the thermal diffusion equation, driven by a thermoelastic source term, which for the assumed one-dimensional heat flow can be cast in the form [2]
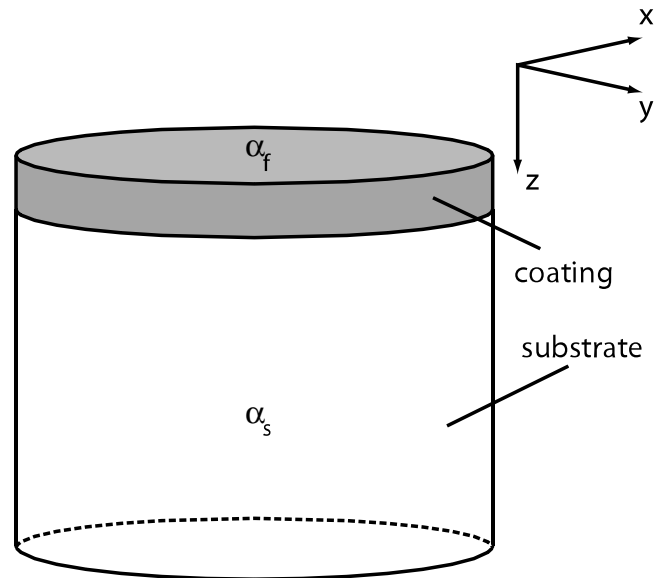

FIG. 1. Diagram showing substrate with multilayer coating where the properties of the coating have been approximated to those of a single layer coating. Shown for clarity are the linear thermal expansion coefficients of the film, α_f and substrate α_s.

$$i\omega\theta_j - \kappa_j \frac{d^2\theta_j}{dz^2} = -\frac{E_j\alpha_j T}{(1-2\nu_j)C_j} i\omega \sum_{i=1}^{3} \varepsilon_{0,ii,j} \quad (1)$$

where $\theta_j(z)$ is the time-varying temperature with $\exp(i\omega t)$ time dependence assumed, $\kappa_j = k_j/C_j$ is the thermal diffusivity, E_j is the Young's modulus, ν_j is the Poisson ratio, T is the background temperature, C_j is the heat capacity per unit volume, $\varepsilon_{0,ii,j}$ is the zeroth-order i-polarized compressional strain, and $j = f, s$ indicates quantities evaluated in the film and the substrate, respectively.

To obtain the source term, we need the zeroth-order compressional strains. Different combinations of zeroth-order elastic fields are relevant in different situations. We will assume that the transverse variation of the zeroth-order elastic fields is slow compared to the thickness of the film, so their only possible variation is in the z-direction, and that variation results only from the possible discontinuity of the elastic properties at the film-substrate boundary. Note that this statement regarding the z-dependence applies only to the zeroth-order elastic fields; as we will see, the thermal fields and the first-order elastic fields they generate have a z-dependence that arises from the propagation of the oscillatory thermal wave itself. Under these assumptions, we can specify the zeroth-order fields in terms of three quantities that, due to the elastic boundary conditions, do not vary over the length scales relevant to this problem: the axial stress $\sigma_0 \equiv \sigma_{0,zz}$, the symmetric combination of in-plane strains (the dilation) $\varepsilon_0 \equiv (\varepsilon_{0,xx} + \varepsilon_{0,yy})/2$, and the anti-symmetric combination of in-plane strains $\varepsilon_{0,xx} - \varepsilon_{0,yy}$. All of the other components of the zeroth-order elastic fields can be derived from these three, as is established in Appendix A. The antisymmetric combination of strains, which is a pure shear along axes rotated $\pi/4$ to x and y, does not interact thermoelastically, and can be neglected in the remainder of the analysis, as is established in more detail in Sec. IV C 1.

Given these zeroth-order elastic fields, we can evaluate the source term in Eq. (1) and solve for the oscillatory thermal wave, $\theta(z)$, as discussed in Sec. IV A and Appendix B. This thermal wave, in turn, generates a first-order elastic field, with compressional components $\varepsilon_{1,ii}$ and $\sigma_{1,ii}$. The thermoelastic coupling enters into the formulation through the elastic equilibrium equations and modified Hooke's law, which can be cast for the one-dimensional case considered here from Eqs. 7.8 and 6.2 of [10],

$$\frac{d}{dz}[\varepsilon_{1,xx,j} + \varepsilon_{1,yy,j} + 2(1-\nu_j)\varepsilon_{1,zz,j} - 2(1+\nu_j)\alpha\theta_j] = 0 \quad (2)$$

and

$$\sigma_{1,ii,j} = \frac{E_j}{1+\nu_j}\left[\varepsilon_{1,ii,j} + \frac{\nu_j}{1-2\nu_j}(\varepsilon_{1,xx,j} + \varepsilon_{1,yy,j} + \varepsilon_{1,zz,j})\right] - \frac{E_j\alpha_j\theta_j}{1-2\nu_j}, \quad (3)$$

where $j = f, s$ represents fields and material properties in the film and substrate, respectively. With the thermal fields obtained by solving Eq. (1), we can obtain from Eqs. (2) and (3) the first-order elastic fields, as given in Appendix C.

The rate at which work is done per unit volume in a deformed body is in general given by

$$p = \sigma_{ik}\frac{d\varepsilon_{ik}}{dt}$$

and the dissipated power density by

$$p_{\text{diss}} \approx -\frac{\omega}{2}\sum_{i=1}^{3} \text{Im}[\sigma_{0,ii}^{*}\varepsilon_{1,ii} + \sigma_{1,ii}^{*}\varepsilon_{0,ii}]. \quad (4)$$

where the second form is specified to our problem with sinusoidal fields and only longitudinal strains, and takes into account that the zeroth-order fields are real, so that dissipation first appears in the product of first- and zeroth-order fields. Integrating the dissipated power density over $0 \leq z < \infty$, we obtain the dissipated power per unit area, given in Sec. IV B. While this is the essential quantity of interest, it is convenient for comparison to experimental measurements of elastic Q to define an effective dissipation factor, ϕ. Since the thermoelastic dissipation is non-local, the choice of stored energy with which to make such a definition is somewhat arbitrary. A reasonable choice, and the one that results in a value for ϕ directly comparable to that derived from experimental results, is the elastic energy stored in the film. With this choice, as described in Sec. IV C 1, we find that for an elastic field with specified in-plane strain and vanishing axial stress, as would be appropriate for a measurement of the elastic Q of a mass coated with a uniform film of thickness l, the total thermoelastic loss, ϕ_{tot} [Eq. (55)] is well approximated by

$$\phi_{l,\parallel} = \frac{2E_f\alpha_f^2 T}{C_f(1-\nu_f)}\left[1 - \frac{\alpha_s}{\alpha_f}\frac{E_s(1-\nu_f)}{E_f(1-\nu_s)}\frac{C_f}{C_s}\right]^2 g(\omega) \quad (5)$$

where the frequency dependence is contained in the function $g(\omega)$ defined by

$$g(\omega) \equiv \text{Im}\left[-\frac{1}{\sqrt{i\omega\tau_f}}\frac{\sinh(\sqrt{i\omega\tau_f})}{\cosh(\sqrt{i\omega\tau_f}) + R\sinh(\sqrt{i\omega\tau_f})}\right], \quad (6)$$

where $\omega = 2\pi f$, $\tau_f \equiv l^2/\kappa_f = l^2 C_f/k_f$ is the thermal diffusion time across the film, and $R \equiv \sqrt{k_f C_f/k_s C_s}$,

with k_j and C_j the thermal conductivity and volumetric heat capacity, respectively. Eqs. (5) and (6) [or the form for a multilayer in Eq. (9)] constitute one of the two key results of this paper. The frequency dependence represented by $g(\omega)$ is discussed at length in Sec. IV C 3. The quantitative implications for measurements of thermoelastic dissipation in several material systems are discussed in Sec. III A.

These numerical results indicate that thermoelastic losses associated with the coating are comparable to those obtained in experimental measurements of elastic loss, which suggests that their contribution to the total displacement noise budget for a test mass could be significant. While one could form an expression for the thermal noise imposed on a Gaussian beam interrogating a coated test mass by inserting $\phi_{l,\parallel}$ from Eq. (5), and a corresponding one for $\phi_{l,\perp}$ from Eq. (41) into one of the expressions developed for thermal noise in coated test masses [6][10], the result would only be approximate because of the thermoelastic coupling between in-plane and normal strains. A direct calculation of the thermal noise can instead be carried out using Levin's formulation, calculating the power dissipated by the thermoelastic mechanism when a pressure field with the same radial distribution as the optical intensity field is applied to the coated mass. This calculation, using the zeroth-order elastic fields obtained in [6], is carried out for arbitrary frequency in Sec. D. We find

$$S_x(f)df = \frac{8k_BT^2}{\pi^2 f}\frac{l}{w^2}\frac{\alpha_s^2 C_f}{C_s^2}(1+\nu_s)^2\Delta^2 g(\omega)df$$
$$\rightarrow \frac{8\sqrt{2}k_BT^2}{\pi\sqrt{\omega}}\frac{l^2}{w^2}(1+\nu_s)^2\frac{C_f^2}{C_s^2}\frac{\alpha_s^2}{\sqrt{k_sC_s}}\Delta^2 df, \quad (7)$$

where Δ^2 is a dimensionless positive-definite combination of material constants that vanishes when the film and substrate are identical,

$$\Delta^2 \equiv \left\{\frac{C_s}{2\alpha_sC_f}\frac{\alpha_f}{(1-\nu_f)}\left[\frac{1+\nu_f}{1+\nu_s}+(1-2\nu_s)\frac{E_f}{E_s}\right]-1\right\}^2, \quad (8)$$

$g(\omega)$ is the same frequency dependence as defined in Eq. (6), and the second form holds for low frequencies obeying $\omega < 1/\tau_f$. Note that the results in Sec. III B show that the limiting form must be used in the gravitational wave detection band only with caution. Equation (7) [or the corresponding Eq. (10) for a multilayer] is the other key result of this paper; quantitative implications for several plausible mass/coating combinations are presented in Sec. III B.

While the results for $\phi_{l,\parallel}$ and $S_x(f)$ in Eqs. (5) and (7) are calculated for a uniform film, most optical coatings of course will consist of a large number of layers. In such a multilayered coating, there are two thermoelastic dissipation peaks, one at a frequency related to the thermal diffusion time through the entire film, and one at a

frequency related to the thermal diffusion time through an individual layer. These interlayer effects are investigated in Sec. IV C 2. It is seen there together with Sec. IV C 3 that for problems of interest, the thermoelastic effects are dominated by contributions from the thermal diffusion through the film, so that a description of the multilayer in terms of a set of averaged properties appears appropriate. The subtleties of the averaging process are investigated in Appendix D, where it is seen that the average of various products of material quantities is required in addition to the average of the quantities themselves. Specializing to a periodic multilayer of total thickness l with N alternating layers of materials a and b in thicknesses of d_a and d_b, the result for $\phi_{l,\parallel}$ from Eq. (42),

$$\phi_{l,\parallel} = \frac{2C_FT}{\left(\frac{E}{1-\nu}\right)_{\text{avg}}}\left[\frac{1}{C_F}\left(\frac{E\alpha}{1-\nu}\right)_{\text{avg}}-\frac{1}{C_s}\frac{E_s\alpha_s}{1-\nu_s}\right]^2 g(\omega) \quad (9)$$

where the frequency dependence is contained in the same function $g(\omega)$ as defined in Eq. (6) with $\tau_f \rightarrow \tau_F$ where $\tau_F = l^2/\kappa_F$, and the result for $S_x(f)$ from Eq. (74),

$$S_x(f) = \frac{8k_BT^2}{\pi^2 f}\frac{l}{w^2}\frac{\alpha_s^2 C_F}{C_s^2}(1+\nu_s)^2\tilde{\Delta}^2 g(\omega)$$
$$\rightarrow \frac{8\sqrt{2}k_BT^2}{\pi\sqrt{\omega}}\frac{l^2}{w^2}(1+\nu_s)^2\frac{C_F^2}{C_s^2}\frac{\alpha_s^2}{\sqrt{k_sC_s}}\tilde{\Delta}^2, \quad (10)$$

and Eq. (73)

$$\tilde{\Delta}^2 \equiv \left\{\frac{C_s}{2\alpha_sC_F}\left(\frac{\alpha}{1-\nu}\left[\frac{1+\nu}{1+\nu_s}+(1-2\nu_s)\frac{E}{E_s}\right]\right)_{\text{avg}}-1\right\}^2, \quad (11)$$

can be stated in terms of an averaging operator defined in Eq. (D1) as

$$(X)_{\text{avg}} \equiv \frac{d_a}{d_a+d_b}X_a + \frac{d_b}{d_a+d_b}X_b, \quad (12)$$

and volume-averaged material properties C_F and κ_F defined in Eq. (D5). The second form of Eq. (10) is, like the second form of Eq. (7), a low-frequency limit valid for $\omega < 1/\tau_f$.

III. NUMERICAL RESULTS

A. Numerical results for dissipation in Q measurements

The mechanical loss factors (or equivalently Q factors) of dielectric coatings applied to test-mass substrates may be obtained experimentally. In a typical measurement of this type, a subset of the vibrational resonant modes of a coated substrate are individually excited above some background level and the decay of the amplitude of vibration of the face of the sample measured as a function of time. From this measurement, and a model of the distribution of the stored energy in each mode of a coated

sample, the mechanical loss factors of the dielectric coating at each of the resonant frequencies of the sample may be obtained.

As part of the loss measurement process described above, the coated samples experience periodic strains. If there exists a difference in the thermoelastic properties of the dielectric coating and the substrate, then as shown in this paper, there will be thermoelastic dissipation. Equation (9) can be used to calculate the thermoelastic dissipation in a coating both at the frequencies typical of mechanical loss measurements, and at frequencies of interest for gravitational wave detection.

The expected thermoelastic loss associated with a coating on a substrate is a direct function of the material parameters for the particular substrates and coatings chosen. Current interferometric detectors use fused-silica substrates with coatings formed from alternating layers of SiO_2 (refractive index $n = 1.45$) and Ta_2O_5 (refractive index $n = 2.03$), each layer being of $\lambda/4$ optical thickness, with $\lambda = 1064$ nm. The mirrors in future upgrades to current detectors are expected to have substrates of either fused silica or sapphire. The choice of appropriate mirror coating materials is a subject of ongoing study [5,6,11], with the two coatings currently under most intense study being alternating layers of SiO_2 and Ta_2O_5, or Al_2O_3 (refractive index $n = 1.63$) and Ta_2O_5.

To estimate the expected level of thermoelastic loss for the mirror/substrate coating combinations above, Eq. (9) was used. In each case a coating thickness equivalent to 30 alternating quarter-lambda layers of the coating materials was chosen. The numerical values used for the properties of the mirror substrates are shown in Tables I and II.

Choosing appropriate material parameters for the multilayer ion-beam-sputtered dielectric coatings is made more difficult by the fact that thermophysical properties of these types of coatings are not well characterized. Absent better information, the properties of the amorphous SiO_2 and Al_2O_3 present in the films were assumed to be the same as the bulk values for amorphous fused silica and crystalline sapphire. The numerical values used for the properties of Ta_2O_5 are summarized in Table III.

Values for Young's modulus and density of Ta_2O_5 in thin film form were taken from Ref. [12]. The coefficient of thermal expansion for Ta_2O_5 film was taken from [13] and a value for the specific heat capacity of Ta_2O_5 (bulk) from [14]. No values for thermal conductivity or Poisson's ratio were available for Ta_2O_5; absent better information we take these to be closer to those of sapphire than silica, and assign them the same values as used for Al_2O_3. Using Eq. (10) the thermoelastic losses from coatings of either SiO_2/Ta_2O_5 or Al_2O_3/Ta_2O_5 applied to silica and sapphire substrates were calculated for frequencies up to 100 kHz, a typical range of interest for measurements of mechanical loss. The estimated loss factors are plotted in Fig. 2.

If the material parameters given above are considered representative then it can be seen from Fig. 2 that in general, the calculated magnitude of the thermoelastic losses from these mirror and coating combinations can be of the order of a few 10^{-6} to approximately 10^{-3}, comparable to the levels of coating loss factors predicted to be significant in estimations of the thermal-noise level in advanced gravitational wave detectors [5,6].

In previous work [11], measurements have been made of the mechanical loss in the frequency range from ~ 2.8 kHz to ~ 73 kHz, for dielectric coatings of SiO_2/Ta_2O_5 applied to fused-silica substrates. The measured coating loss factors were found to be of the order of 2.8×10^{-4}. It can be seen from Fig. 2, (curve a) that the estimated thermoelastic losses for this particular coating/substrate combination are much smaller than the measured losses. This suggests that the measured losses are not predominantly thermoelastic in origin, and are associated with some other form of dissipation. However in this case the calculated loss factor depends strongly on the square of the linear expansivity of the film. An increase in the expansivity of Ta_2O_5 in thin film form by a factor of approximately three over the value taken here would increase the predicted thermoelastic loss to a level comparable with that measured experimentally. However, such an increased expansion coefficient would be considerably higher than found in recent measurements by Braginsky and Samoilenko [8].

TABLE I.   Fused-silica substrate properties [16] [17].

$\alpha_s = 5.1 \times 10^{-7}$ K^{-1}
$E_s = 7.2 \times 10^{10}$ Nm^{-2}
$C_s = 746$ Jkg^{-1} K$^{-1} \times 2200$ kgm^{-3}
$\quad = 1.64 \times 10^6$ JK^{-1} m^{-3}
$k_s = 1.38$ Wm^{-1} K^{-1}
$\nu_s = 0.17$

TABLE II.   Sapphire substrate properties [16] [17] .

$\alpha_s = 5.4 \times 10^{-6}$ K^{-1}
$E_s = 4 \times 10^{11}$ Nm^{-2}
$C_s = 777$ Jkg^{-1} K$^{-1} \times 3980$ kgm^{-3}
$\quad = 3.09 \times 10^6$ JK^{-1} m^{-3}
$k_s = 33$ Wm^{-1} K^{-1}
$\nu_s = 0.23$

TABLE III.   Properties used for Ta_2O_5 in thin film form.

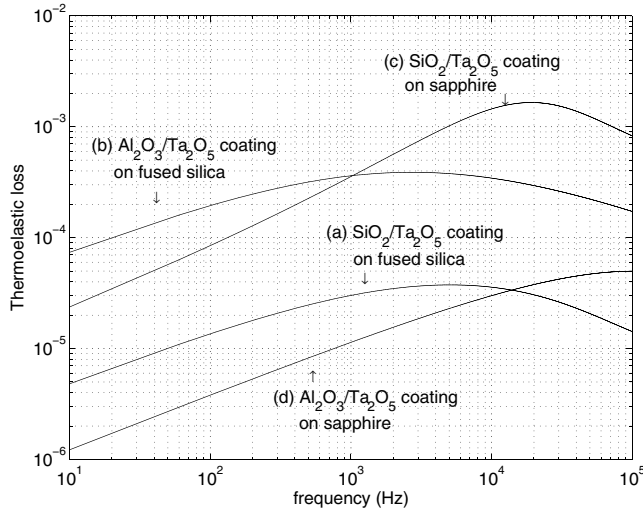| |
|---|
| $\alpha = 3.6 \times 10^{-6}$ K^{-1} |
| $E = 1.4 \times 10^{11}$ Nm^{-2} |
| $C = 306$ Jkg^{-1}K$^{-1} \times 6850$ kgm^{-3} |
| $\quad = 2.1 \times 106$ JK^{-1} m^{-3} |
| $k = 33$ Wm^{-1} K^{-1} |
| $\nu = 0.23$ |

FIG. 2. Calculated frequency dependence of the thermoelastic losses of fused-silica substrates with dielectric multilayer coatings formed from 30 alternating quarter-lambda layers of (a) SiO_2 and Ta_2O_5, (b) Al_2O_3 and Ta_2O_5, (c) SiO_2 and Ta_2O_5 or (d) Al_2O_3 and Ta_2O_5. Shown are loss factors for frequencies up to 100 kHz.

The thermoelastic losses for other combinations of mirror and coating materials are estimated to be considerably larger than is the case for SiO_2/Ta_2O_5 coatings on silica substrates, see, for example, Fig. 2, (curve b) for an Al_2O_3/Ta_2O_5 coating on a silica substrate. Available data on material properties suggests that in this case thermoelastic loss could account for a larger fraction of measured dissipation but the currently available experimental data is sufficiently inconclusive to allow quantitative comparison here. Since this form of dissipation is frequency dependent, it is clearly important to estimate the magnitude of the dissipation in the frequency range of interest for gravitational wave detectors.

From Fig. 2 it can be seen that the thermoelastic loss in the gravitational wave detection band, ~ 10 Hz to ~ 1 kHz, is lower than at the higher frequencies sampled by mechanical loss measurements, however it can still be of the order of 10^{-4}. It should be noted that in the absence of dissipation from other sources thermal noise arising from coating-related thermoelastic losses will form a limit to the thermal-noise performance of interferometric detectors in a manner similar to the thermoelastic noise from the substrates themselves [3]. Section D thus addresses the derivation of an expression for the thermal noise from coatings arising from thermoelastic dissipation, numerical results from which are presented in Sec. III B.

B. Thermal noise

Using Eqs. (10) and (72) with the parameters for coating and substrate properties given earlier, the thermal displacement noise resulting from thermoelastic dissipa-

tion, $\sqrt{S_x(f)}$, associated with silica and sapphire mirrors with coatings of either SiO_2/Ta_2O_5 or Al_2O_3/Ta_2O_5 can be estimated. Here, multilayer coatings of a thickness equivalent to 10 ppm transmission were modeled, since this represents a typical specification for the transmission of a mirror coating used in the Fabry-Perot arm cavities of a gravitational wave detector. Figure 3 shows the calculated noise for each case, for frequencies up to 1 kHz. A beam radius, *w*, of 5.5 cm was assumed.

For comparison, the target level for total displacement noise per test mass in the advanced LIGO gravitational wave interferometer design is approximately 6×10^{-21} m/$\sqrt{Hz}$ at 100 Hz if sapphire mirrors are used and approximately 8×10^{-21} m/$\sqrt{Hz}$ for silica mirrors. [15] Fig. 3 shows that for SiO_2/Ta_2O_5 coatings on silica or sapphire substrates and Al_2O_3/Ta_2O_5 coatings on sapphire substrates, the expected coating-related thermoelastic displacement noise is below the required specification at 100 Hz. However for an Al_2O_3/Ta_2O_5 coating on a fused-silica substrate the noise from this dissipation mechanism alone is above the specification at 100 Hz. It is also clear that the same coating will result in a different level of noise if applied to different substrates.

There are several other points illustrated by Fig. 3 worth consideration. Firstly, for some coating/substrate combinations the thermoelastic noise starts to deviate significantly from the explicit low-frequency limit in the frequency range of interest for detector operation. Thus when estimating the expected level of this noise it
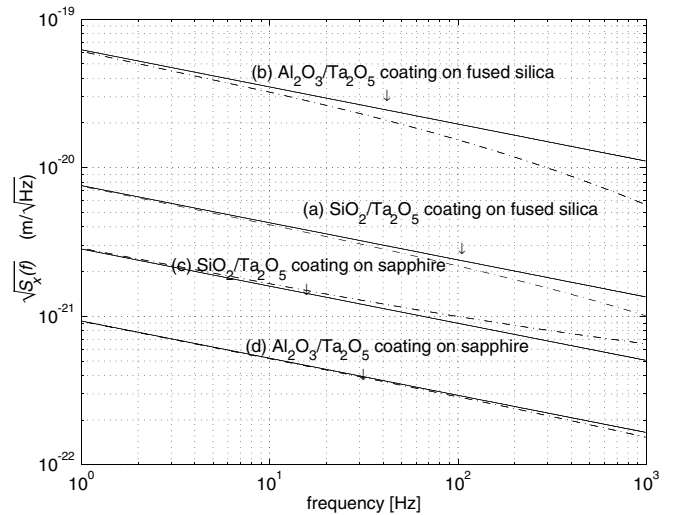

FIG. 3. Calculated thermoelastic thermal noise, $\sqrt{S_x(f)}$ for coatings of (a) an SiO_2/Ta_2O_5 coating applied to a silica substrate (b) an Al_2O_3/Ta_2O_5 applied to a silica substrate (c) an SiO_2/Ta_2O_5 coating applied to a sapphire substrate and (d) an Al_2O_3/Ta_2O_5 coating applied to a silica substrate. Shown in each case is the noise obtained using the explicit low-frequency limit for the noise (solid line), and the noise including the full frequency dependence (dashed line), as given in Eq. (10).

is important to use the full frequency-dependent expression. In addition, comparing Figs. 2 and 3 it can be seen that while the thermoelastic loss for strain fields associated with typical loss measurements, $\phi_{l,\parallel}$ is higher for a SiO_2/Ta_2O_5 coating on a sapphire substrate than for that coating on a silica substrate, the opposite trend holds for the thermoelastic displacement noise sensed in an interferometer. This seeming contradiction can be understood by comparing Eqs. (37) for $\phi_{l,\parallel}$ and (41) for $\phi_{l,\perp}$, the loss for a specified surface-normal stress. We see that the dependence of these two loss coefficients on the material properties is quite different, in particular, containing a ratio of Young's moduli in the former but not the latter. In fact, for the material properties characteristic of these coatings, unlike $\phi_{l,\parallel}$, $\phi_{l,\perp}$ follows the same trend as S_x, consistent with the observation that the axial stress is large in the region of high optical intensity for the fields of Eq. (66) used in the noise calculation.

It is also important to note that in each case the exact level of thermoelastic noise is a strong function of certain of the material parameters of the coatings, in particular, the coefficient of thermal expansion, and given the lack of information available on the thermoelastic properties of ion-beam-sputtered coatings, our calculations here were carried out using plausible rather than definitive values for relevant coating material parameters. Thus these figures should be taken as estimates of the expected thermoelastic noise due to the coatings, rather than reliable results that can be used in design calculations.

IV. DETAILED CALCULATION

A. Thermal fields

We start by calculating the thermal field $\theta(z, t)$ generated by the applied zeroth-order elastic fields from Appendix A. We can cast the one-dimensional heat equation in the form [2]

$$\frac{\partial \theta_j}{\partial t} - \kappa_j \frac{\partial^2 \theta_j}{\partial z^2} = -\frac{E_j \alpha_j T}{(1 - 2\nu_j)C_j} \frac{\partial}{\partial t} \sum_{i=1}^{3} \varepsilon_{0,ii,j} \quad (13)$$

where θ_j is the time-varying temperature, $\kappa_j = k_j/C_j$ is the thermal diffusivity, E_j is the Young's modulus, ν_j is the Poisson ratio, T is the background temperature, C_j is the heat capacity per unit volume, $\varepsilon_{0,ii,j}$ is the zeroth-order i-polarized compressional strain, and $j = f, s$ indicates quantities evaluated in the film and the substrate, respectively. Taking sinusoidally time-varying quantities according to $a(z, t) = \text{Re}[a(z) \exp(i\omega t)]$, Eq. (13) becomes

$$i\omega \theta_j(z) - \kappa_j \frac{\partial^2 \theta_j(z)}{\partial z^2} = -i\omega \beta_j, \quad (14)$$

where the source term is proportional to

$$\beta_j \equiv \frac{E_j \alpha_j T}{C_j} \frac{\Sigma_j}{1 - 2\nu_j}. \quad (15)$$

The quantity Σ_j representing the sum of the strains according to

$$\Sigma_j \equiv \sum_{i=1}^{3} \varepsilon_{0,ii,j}, \quad (16)$$

is proportional to the zeroth-order elastic field's amplitude with a combination of elastic constants that depends on the specific case, and is evaluated in Appendix A. α_f is possibly z-dependent to allow for spatially varying thermal expansion coefficient within the film. The boundary conditions are zero heat flux at $z = 0$, continuity of heat flux at $z = l$, and vanishing heat flux for $z \rightarrow \infty$, or

$$\frac{d\theta_f}{dz}\Big|_{z=0} = 0, k_f \frac{d\theta_f}{dz}\Big|_{z=l} = k_s \frac{d\theta_s}{dz}\Big|_{z=l}, \frac{d\theta_s}{dz}\Big|_{z\rightarrow\infty} = 0, \quad (17)$$

respectively.

The total solution can be constructed as the sum of homogeneous and particular solutions. Homogeneous solutions meeting the boundary conditions at $z = 0$ and $z \rightarrow \infty$ are

$$\theta_{h,f}(z) = \theta_{1f} \cosh(\gamma_f z) \quad \text{and} \quad \theta_{h,s}(z) = \theta_{1s} e^{-\gamma_s z} \quad (18)$$

where the complex propagation constants of the damped thermal waves in the film and substrate are

$$\gamma_j \equiv (1 + i)\sqrt{\omega/2\kappa_j} \quad (19)$$

and θ_{1f} and θ_{1s} are constants determined by the boundary condition at $z = l$ and the particular solution.

The particular solutions will be evaluated for two specific cases of practical interest in Appendix B. For the time being take them to be $\theta_{p,j}(z)$, so that the total solutions are

$$\theta_f(z) = \theta_{p,f}(z) + \theta_{1f} \cosh(\gamma_f z), \quad \theta_s(z) = \theta_{p,s} + \theta_{1s} e^{-\gamma_s z} \quad (20)$$

Note that both $\theta_{p,j}(z)$ and $\theta_{1,j}$ will be proportional to the amplitude of the zeroth-order elastic fields.

B. Elastic fields and energy density

The rate at which work is done per unit volume by internal stresses on a deformed body is [10]

$$p = \sigma_{ik} \frac{d\varepsilon_{ik}}{dt}. \quad (21)$$

This expression is correct independent of whether the body responds elastically or anelastically to the stresses. The cycle average of the delivered power density (or, equivalently, the average dissipated power density) is then, for fields sinusoidal in time of the form $\sigma_{ii,j}(z, t) = \text{Re}\{[\sigma_{0,ii,j} + \sigma_{1,ii,j}(z)] \exp(i\omega t)\}$ and similar for $\varepsilon_{ii,j}(z, t)$,

$$p_{\text{diss}}(z) = -\frac{\omega}{2} \sum_{i=1}^{3} \text{Im}[\sigma_{ik}^* \varepsilon_{ik}]$$

$$= -\frac{\omega}{2} \sum_{i=1}^{3} \text{Im}[\sigma_{0,ii}^* \varepsilon_{0,ii} + \sigma_{0,ii}^* \varepsilon_{1,ii} + \sigma_{1,ii}^* \varepsilon_{0,ii}$$

$$+ \sigma_{1,ii}^* \varepsilon_{1,ii}]$$

$$\approx -\frac{\omega}{2} \sum_{i=1}^{3} \text{Im}[\sigma_{0,ii}^* \varepsilon_{1,ii} + \sigma_{1,ii}^* \varepsilon_{0,ii}]. \tag{22}$$

The last form of this equation is justified by the following observations: For our problem, only the longitudinal strains are significant and the zeroth-order elastic fields are real [Appendix A, and Eqs. (66)], while the first-order elastic fields [those that depend on the thermal field, Appendix C] are complex and so contribute to the dissipation. We assume that the dissipation is small, so that a calculation to lowest order in the thermal field will be adequate, and second order terms can be dropped.

To evaluate Eq. (22), we need the zeroth- and first-order elastic fields in the film and the substrate, as derived in Appendices A and C. It is convenient to write the zeroth-order fields in terms of σ_0 and ε_0, two combinations of the fields that are invariant through the region of interest in the body. These are defined by $\sigma_0 \equiv \sigma_{0,zz}$, the compressional stress normal to the surface of the object, and $\varepsilon_0 \equiv (\varepsilon_{0,xx} + \varepsilon_{0,yy})/2$, the in-plane dilation. For convenience, we can take $\varepsilon_{0,xx} = \varepsilon_{0,yy}$, though only their sum matters for the thermoelastic calculation. The antisymmetric combination of in-plane strains is a pure shear, does not interact thermoelastically, and so be can neglected in this analysis (as is discussed in Sec. IV C 1). Note that in cylindrical coordinates $\varepsilon_0 = (\varepsilon_{0,rr} + \varepsilon_{0,\theta\theta})/2$. The zeroth-order fields can then be summarized as

$$\varepsilon_{0,ii,j} = A_{0,ii,j} \varepsilon_0 + a_{0,ii,j} \sigma_0, \quad \sigma_{0,ii,j} = B_{0,ii,j} \varepsilon_0 + b_{0,ii,j} \sigma_0, \tag{23}$$

where $j = f, s$, and the ($A_{0,ii,j}$ and $B_{0,ii,j}$) and the ($a_{0,ii,j}$ and $b_{0,ii,j}$) are combinations of elastic constants given in Eqs. (A7) and (A14), respectively. Similarly, it is convenient to write the first-order elastic fields as proportional to the local temperature and thermal expansion coefficient, since we assume that the frequencies of interest are low enough that the elastic response can be treated quasistatically:

$$\varepsilon_{1,ii,j}(z) = A_{1,ii,j} \alpha_j \theta_j(z), \quad \sigma_{1,ii,j}(z) = B_{1,ii,j} \alpha_j \theta_j(z) \tag{24}$$

The (real) coefficients $A_{1,ii,j}$ and $B_{1,ii,j}$ are given in Eqs. (C7).

With Eqs. (23) and (24) in Eq. (22), the dissipated power density can be written as

$$p_{\text{diss},j}(z) = \omega \frac{\alpha_j}{2} (D_j \varepsilon_0 + d_j \sigma_0) \text{Im}[-\theta_j(z)] \tag{25}$$

where

$$D_j \equiv \sum_{i=1}^{3} [B_{0,ii,j} A_{1,ii,j} - B_{1,ii,j} A_{0,ii,j}]$$

$$d_j \equiv \sum_{i=1}^{3} [b_{0,ii,j} A_{1,ii,j} - B_{1,ii,j} a_{0,ii,j}]. \tag{26}$$

The dissipated power per unit area is then obtained by integrating the power density over the thickness of the body,

$$\frac{P_{\text{diss}}}{\text{area}} = \int_0^\infty p_{\text{diss}}(z) dz$$

$$= \frac{\omega}{2} \left\{ (D_f \varepsilon_0 + d_f \sigma_0) \int_0^l \alpha_f(z) \text{Im}[-\theta_f(z)] dz \right.$$

$$\left. + (D_s \varepsilon_0 + d_s \sigma_0) \alpha_s \int_l^\infty \text{Im}[-\theta_s(z)] dz \right\} \tag{27}$$

where we allow for the possibility of spatial variation in the thermal expansion coefficient in the film but assume that it is uniform in the substrate.

We can use this expression both to evaluate the dissipation that would be measured in a typical measurement of the Q of a coated test mass and to calculate the coating-related thermoelastic contribution to the displacement noise imposed on an optical field incident on a test mass. The most convenient form of the analysis is somewhat different in these two contexts. We begin with the case of a Q measurement.

C. Effective thermoelastic losses in measurements of elastic Q

While the total dissipated power given in Eq. (27) is the physical quantity of importance to measurements of Q, and is nonlocal in nature, occurring in both the film and the substrate, it generally occurs in a region thin compared to the dimensions of the test mass, so for comparison with experimental results it is convenient to describe the loss in terms of an effective ϕ associated with the coating. To define such an effective ϕ, we must compare the dissipated power to some stored energy. A reasonable choice of stored energy for the definition of ϕ is that in the film, i.e.

$$U_{\text{stor}}/\text{area} = \frac{l}{2} \sum_{i=1}^{3} \text{Re}[\sigma_{0,ii,f}^* \varepsilon_{0,ii,f}] = l \frac{|\varepsilon_0^2|}{2} U_f, \tag{28}$$

where

$$U_f \equiv \sum_{i=1}^{3} B_{0,ii,f} A_{0,ii,f}. \tag{29}$$

In writing Eqs. (28) and (29) we assumed that the film is on a stress-free surface, so that $\sigma_{0,zz}$ and hence σ_0 vanish. U_f is calculated in Appendix A, Eq. (A9). We then have

for ϕ, with Eq. (27)

$$\phi = \frac{P_{\text{diss}}\tau}{2\pi U_{\text{stor}}} \qquad = \phi_f + \phi_s \qquad (30)$$

where

$$\phi_f \equiv \frac{D_f}{U_f l} \int_0^l \alpha_f(z)\text{Im}[-\theta_f(z)/\varepsilon_0]dz \qquad (31)$$

and

$$\phi_s \equiv \frac{D_s}{U_f l}\alpha_s \int_l^\infty \text{Im}[-\theta_s(z)/\varepsilon_0]dz. \qquad (32)$$

Note that since the thermal fields are proportional to ε_0, the quantity in square brackets in Eqs. (31) and (32) is independent of ε_0, as are all the other factors in these equations.

To make further progress, we must find the particular solution and the coefficients for the homogeneous solutions for the thermal field in the specific cases of interest. We consider the specific cases of a uniform film on a uniform substrate, and a periodic film on a uniform substrate. In Appendix B we obtain the thermal fields for these two cases.

1. Uniform film and substrate

Consider first the simple model of a uniform film on a substrate, with possibly different thermophysical properties in film and substrate. For this case, we can take the thermal expansion coefficients in the film and substrate to be $\alpha_f(z) = \alpha_f$ and α_s, respectively, and the particular solutions to the heat equation, given in Appendix B as Eq. (B5), are

$$\theta_{p,j}(z) = -\beta_j. \qquad (33)$$

Since the particular solutions are real, the only contribution to the imaginary part of the integrals in Eqs. (31) and (32) come from the homogeneous solutions, so with Eqs. (18) we have for the film,

$$\begin{aligned}
\phi_f &= \frac{D_f}{U_f l} \int_0^l \alpha_f(z)\text{Im}[-\theta_f(z)/\varepsilon_0]dz \\
&= \frac{D_f\alpha_f}{U_f l}\text{Im}\left[-(\theta_{1,f}/\varepsilon_0) \int_0^l \cosh(\gamma_f z)dz\right] \\
&= \frac{D_f\alpha_f}{U_f l}\text{Im}[-(\theta_{1,f}/\varepsilon_0)\gamma_f^{-1}\sinh(\gamma_f l)] \qquad (34)
\end{aligned}$$

and similarly for the substrate

$$\phi_s = \frac{D_s\alpha_s}{U_f l}\text{Im}[-(\theta_{1,s}/\varepsilon_0)\gamma_s^{-1}e^{-\gamma_s l}]. \qquad (35)$$

where the D_j are combinations of elastic constants defined in Eq. (26) and calculated in Eq. (C8). Summing the contributions to the dissipation from the film and the substrate, Eqs. (34) and (35), we can express the total

loss as

$$\begin{aligned}
\phi_{l,\parallel} &= \frac{D_f\alpha_f}{U_f l}\text{Im}[-(\theta_{1,f}/\varepsilon_0)\gamma_f^{-1}\sinh(\gamma_f l)] \\
&\quad + \frac{D_s\alpha_s}{U_f l}\text{Im}[-(\theta_{1,s}/\varepsilon_0)\gamma_s^{-1}e^{-\gamma_s l}], \qquad (36)
\end{aligned}$$

where the subscript l is used to indicate a quantity resulting from thermoelastic behavior over the entire thickness of the film (contrasted to multilayer case in the following section) and $\parallel$ indicates the case of specified in-plane strain (in contrast to $\perp$ for specified surface-normal stress). With the coefficients $\theta_{1,j}$ from Appendix B Eqs. (B6) and (B7) inserted into Eq. (36), we can express the total loss, after some algebra, as

$$\begin{aligned}
\phi_{l,\parallel} &= \frac{\Delta\beta}{\varepsilon_0}\frac{C_f(1-\nu_f)}{E_f}\left[\frac{\alpha_f E_f}{1-\nu_f} - \frac{\alpha_s E_s}{1-\nu_s}\frac{R\gamma_f}{\gamma_s}\right]g(\omega) \\
&= \frac{2C_f(1-\nu_f)T}{E_f}\left[\frac{\alpha_f E_f}{C_f(1-\nu_f)} - \frac{\alpha_s E_s}{C_s(1-\nu_s)}\right]^2 g(\omega) \\
&= \frac{2E_f\alpha_f^2 T}{C_f(1-\nu_f)}\left[1 - \frac{\alpha_s}{\alpha_f}\frac{E_s(1-\nu_f)}{E_f(1-\nu_s)}\frac{C_f}{C_s}\right]^2 g(\omega) \qquad (37)
\end{aligned}$$

where the frequency dependence is contained in the function $g(\omega)$ defined by

$$g(\omega) \equiv \text{Im}\left[-\frac{1}{\sqrt{i\omega\tau_f}}\frac{\sinh(\sqrt{i\omega\tau_f})}{\cosh(\sqrt{i\omega\tau_f}) + R\sinh(\sqrt{i\omega\tau_f})}\right]. \qquad (38)$$

In deriving this result, we made use of D_j from Eq. (C8), U_f from Eq. (A9), R from Eq. (B4), $\Delta\beta \equiv \beta_f - \beta_s$ from Eq. (15), Σ from Eqs. (A1) and (A8), and defined $\tau_f \equiv l^2/\kappa_f$ so that with Eq. (19) we have $\gamma_f l = \sqrt{i\omega\tau_f}$. Note that the combination of material properties in square brackets in Eq. (37) is positive-definite and vanishes if the film and substrate properties are identical. The quadratic dependence on the *difference* between substrate and film properties can lead to dissipation that is sensitive to small changes in the film properties.

A similar analysis can be carried out for an antisymmetric in-plane strain, $\varepsilon_{0,xx} = -\varepsilon_{0,yy}$. We find that $\Sigma_j = 0$, so that no thermal wave is generated (consistent with the observation that this antisymmetric strain is a pure shear along axes rotated $\pi/4$ with respect to x and y, causes no volume change, and hence does not contribute to the source term for the thermal wave). We also find that $D_j = 0$, indicating that there will be no power dissipated by the interaction of the zeroth-order antisymmetric strain with the first-order strain fields generated by the thermal wave (driven by other zeroth-order strains possibly present). This latter observation can be explained by noting that the thermal wave generates no first-order shear strains (for the geometry considered here), and that there

is no energy term associated with the product of shear and compressional strains in isotropic media.

While not encountered in the measurement of elastic loss in coated masses, an expression for the dissipation for a specified surface-normal stress is useful for developing an understanding of the thermal-noise results in comparison to results for loss measurements. Following the same analysis as was used to find $\phi_{l,\parallel}$, but replacing the stored energy in Eq. (28) with

$$U_{\text{stor}}/\text{area} = l \frac{|\sigma_0^2|}{2} u_f, \tag{39}$$

where

$$u_f \equiv \sum_{i=1}^{3} b_{0,ii,f} a_{0,ii,f}. \tag{40}$$

We find that replacing $U_f \to u_f$, $D_j \to d_j$, and $\varepsilon_0 \to \sigma_0$ in Eq. (36) yields the correct result for $\phi_{l,\perp}$,

$$\phi_{l,\perp} = \frac{E_f \alpha_f^2 T}{C_f} \frac{1 + \nu_f}{(1 - \nu_f)(1 - 2\nu_f)} \left[1 - \frac{\alpha_s}{\alpha_f} \right.$$
$$\left. \times \frac{(1 - \nu_f)(1 + \nu_s)}{(1 - \nu_s)(1 + \nu_f)} \frac{C_f}{C_s} \right]^2 g(\omega), \tag{41}$$

where we made use of d_j from Eq. (C9), u_f from Eq. (A16), Σ from Eqs. (A1) and (A15), other quantities as after Eq. (38), and $g(\omega)$ is the same frequency dependence given in Eq. (38).

2. Modulated film and uniform substrate

Optical coatings of interest for use on test masses for gravitational wave detectors are invariably multilayered, so the analysis in the previous section of a uniform film cannot be correct in detail. Because the thermal diffusion length for frequencies of interest are in general long compared to the thickness of individual layers in the film, an accurate approximation for a multilayer coating can be obtained by using a suitable averaging process to model it as a uniform layer (except at very high frequencies), as discussed in Appendix D. Following the procedure described there, we find that the result in Eq. (37) is replaced by

$$\phi_{l,\parallel} = \frac{2 C_F T}{(\frac{E}{1-\nu})_{\text{avg}}} \left[\frac{1}{C_F} \left(\frac{E\alpha}{1-\nu} \right)_{\text{avg}} - \frac{1}{C_s} \frac{E_s \alpha_s}{1-\nu_s} \right]^2 g(\omega) \tag{42}$$

and the frequency dependence $g(\omega)$ is unchanged except for replacing the time constant τ_f by an appropriately averaged one τ_F. The volume-weighted average indicated by $(X)_{\text{avg}}$ is defined in Eq. (D1), the averaged heat capacity C_F in Eq. (D4), and τ_F in Eq. (D6).

While the results based on this averaging process appear reasonable, it is useful to explore for unexpected effects associated with the spatial variation of thermoelastic properties within the multilayer film. As a simple

model of such a case, we consider a film whose thermal expansion coefficient, but no other property, varies periodically, on a uniform substrate. The calculation is similar in principal, but somewhat more complicated than for the case of a uniform film on a uniform substrate. For this case, we take a thermal expansion coefficient of the form

$$\alpha_f(z) = \overline{\alpha}_f + \alpha_m \cos(K_m z) \quad \text{and} \quad \alpha_s(z) = \alpha_s \tag{43}$$

and the particular solution to the heat equation in the film, given in Appendix B as Eq. (B13) is

$$\theta_{p,f}(z) = -\beta_f - \beta_m \frac{\gamma_f^2}{\gamma_f^2 + K_m^2} \cos(K_m z) \tag{44}$$

while the particular solution in the substrate, Eq. (B14), remains the same as for the uniform film case

$$\theta_{p,s} = -\beta_s. \tag{45}$$

For simplicity, we consider here only the case of specified in-plane strain and vanishing surface-normal stress σ_0. Since the particular solution in the film has a complex part, the imaginary part of the integrals in Eqs. (31) and (32) will contain contributions from both the particular and the homogeneous solutions. With Eqs. (18), (43) and (44) we have in the film

$$\phi_f = \frac{D_f}{U_f l} \int_0^l \alpha_f(z) \text{Im}[-\theta_f(z)/\varepsilon_0] dz$$
$$= \frac{D_f}{U_f l} (\Phi_{f,p} + \Phi_{f,h}), \tag{46}$$

where

$$\Phi_{f,p} \equiv \text{Im}\left[-\int_0^l (\overline{\alpha}_f + \alpha_m \cos[K_m z])(\theta_{p,f}[z]/\varepsilon_0) dz \right]$$
$$= \frac{\beta_m}{K_m \varepsilon_0} \left\{ \overline{\alpha}_f \sin(K_m l) \right.$$
$$\left. + \alpha_m \left[\frac{K_m l}{2} + \frac{\sin(2K_m l)}{4} \right] \right\} \text{Im}\left[\frac{\gamma_f^2}{\gamma_f^2 + K_m^2} \right] \tag{47}$$

and

$$\Phi_{f,h} \equiv \text{Im}\left[-\int_0^l (\overline{\alpha}_f + \alpha_m \cos[K_m z])(\theta_{h,f}[z]/\varepsilon_0) dz \right]$$
$$= -\text{Im}\left[\overline{\alpha}_f \frac{\theta_{1,f}}{\varepsilon_0} \gamma_f^{-1} \sinh\gamma_f l \quad + \alpha_m \frac{\theta_{1,f}}{\varepsilon_0} \right.$$
$$\left. \times \frac{K_m \cosh(\gamma_f l) \sin(K_m l) + \gamma_f \sinh(\gamma_f l) \cos(K_m l)}{K_m^2 + \gamma_f^2} \right] \tag{48}$$

Since the thermal expansion coefficient in the substrate is assumed uniform, the result for ϕ_s is similar to that of Eq. (35) (though of course $\theta_{1,s}$ will be different in the two cases). We use Eqs. (18), (43) and (45) to obtain for the substrate

$$\phi_s = \frac{D_s \alpha_s}{U_f l} \text{Im}[-(\theta_{1,s}/\varepsilon_0)\gamma_s^{-1}e^{-\gamma_s l}]. \tag{49}$$

Considerably more effort is required to convert Eqs. (47), (48), and (49) into a simple form like Eq. (37) after inserting $\theta_{1,f}$ and $\theta_{1,s}$ from Eqs. (B15) and (B16). The following two terms emerge as dominant in Eqs. (47) and (48) for cases where $|K_m| >> |\gamma_f|$,

$$\Phi_{f,p} \approx \frac{\alpha_m l}{2}\frac{\beta_m}{\varepsilon_0}\text{Im}\left[\frac{\gamma_f^2}{\gamma_f^2 + K_m^2}\right] \quad \Phi_{f,h} \approx \overline{\alpha}_f\frac{\Delta\beta}{\varepsilon_0}g(\omega), \tag{50}$$

so that with Eq. (46) we have

$$\phi_f \approx \frac{D_f}{U_f l}\left\{\frac{\alpha_m l}{2}\frac{\beta_m}{\varepsilon_0}\text{Im}\left[\frac{\gamma_f^2}{\gamma_f^2 + K_m^2}\right] + \overline{\alpha}_f\frac{\Delta\beta}{\varepsilon_0}g(\omega)\right\}, \tag{51}$$

where $g(\omega)$ is defined in Eq. (38). Similarly, keeping the leading term in Eq. (49) after inserting Eq. (B16) leads to

$$\phi_s \approx -\frac{\Delta\beta}{\varepsilon_0}\frac{D_s\alpha_s}{U_f l}Rg(\omega) \tag{52}$$

where all neglected terms are smaller by at least one factor of γ_f/K_m than those retained. The first term in Eq. (51) is unique to a modulated film. The second term in Eq. (51) and Eq. (52) are just the same as those that appeared in the expression for a homogeneous film and substrate, Eq. (36) with $\alpha_f \rightarrow \overline{\alpha}_f$, so that portion of the solution can be used here immediately. Rewriting Eqs. (51) and (52) with D_j from Eq. (C8), U_f from Eq. (A9), R from Eq. (B4), β_m from Eq. (B10), β_j from Eq. (15), γ_j from Eq. (19), and Σ_j from Eq. (A1) with Eq. (A8) we obtain

$$\phi_{\text{tot},\parallel} \approx \frac{E_f T \alpha_m^2}{C_f(1-\nu_f)}\text{Im}\left[\frac{\gamma_f^2}{\gamma_f^2 + K_m^2}\right] + \phi_{l,\parallel}, \tag{53}$$

where $\phi_{l,\parallel}$ is the loss for a uniform film on a uniform substrate given in Eq. (37), with $\alpha_f \rightarrow \overline{\alpha}_f$. Defining a characteristic time for diffusion in the modulated structure,

$$\tau_m \equiv \frac{\tau_f}{K_m^2 l^2} = \frac{1}{\kappa_f K_m^2} \tag{54}$$

where the second form follows from the definition of τ_f following Eq. (38), $\phi_{\text{tot},\parallel}$ can be written in the form

$$\phi_{\text{tot},\parallel} \approx \frac{E_f T \alpha_m^2}{C_f(1-\nu_f)}g_m(\omega) + \phi_{l,\parallel}, \tag{55}$$

where the frequency dependence $g_m(\omega)$ is

$$g_m(\omega) = \frac{\omega\tau_m}{1 + \omega^2\tau_m^2}. \tag{56}$$

In Sec. IV C 3 we show that for the frequencies of interest, the frequency dependence $g_m(\omega)$ from Eq. (56)

representing heat flow between the multilayers is, as expected, small compared to $g(\omega)$ from Eq. (38) representing heat flow between film and substrate, so that considering only the contribution of the averaged form $\phi_{l,\parallel}$ from Eq. (42) is a good approximation.

3. Frequency dependence

The functions $g(\omega)$ and $g_m(\omega)$, defined in Eqs. (38) and (56), respectively, contain all the frequency dependence of the dissipation, and will reappear in our discussion of displacement noise in Sec. D. It is therefore worth investigating their general features in some detail. Consider first $g(\omega)$; it is convenient to define the real variable ξ according to

$$\gamma_f l = \sqrt{i\omega\tau_f} \equiv (1+i)\xi/2 \tag{57}$$

so that

$$\xi \equiv \sqrt{\frac{2\omega l^2}{\kappa_f}} = \sqrt{2\omega\tau_f} \tag{58}$$

where from Eq. (38) $\tau_f = l^2 C_f/k_f$. The frequency dependence of the loss due to a uniform film can then be written as

$$g(\xi) = \xi^{-1}\frac{\sinh\xi - \sin\xi + R(\cosh\xi - \cos\xi)}{\cosh\xi + \cos\xi + 2R\sinh\xi + R^2(\cosh\xi - \cos\xi)}. \tag{59}$$

In terms of the normalized frequency ξ, the normalized dissipation depends only on one parameter, R, defined after Eq. (B4) as $R \equiv k_f\gamma_f/k_s\gamma_s = \sqrt{k_f C_f/k_s C_s}$. Figure 4 shows the frequency dependence of the dissipation for $R = (0.1, 1, 10)$. Figure 5 shows the dependence on R of g_{max}, the peak value of the normalized dissipation, and Fig. 6 the dependence on R of ξ_{max}, the normalized frequency at which this peak occurs. We see that g_{max} depends only weakly on R, ranging from 0.41 for $R \ll 1$ to 0.21 for $R \gg 1$. ξ_{max} is close to the thermal diffusion time across the film ($\xi = 2$) for $R \leq 1$, and decreases as $1/R$ with R for $R > 1$.

Useful forms for low- and high-frequency limits of the dissipation can be obtained from Eq. (59). Expanding for $\xi \ll 1$ we find

$$g(\xi) \rightarrow \frac{1}{2}\left[R\xi - \left(R^2 - \frac{1}{3}\right)\xi^2\right] \tag{60}$$

while for $\xi \gg 1$

$$g(\xi) \rightarrow \frac{1}{(1+R)\xi}. \tag{61}$$

Since $\omega \propto \xi^2$, the leading behavior for low frequencies goes as the $\sqrt{\omega}$, and, surprisingly, the sign of the term linear in frequency depends on the value of R, crossing
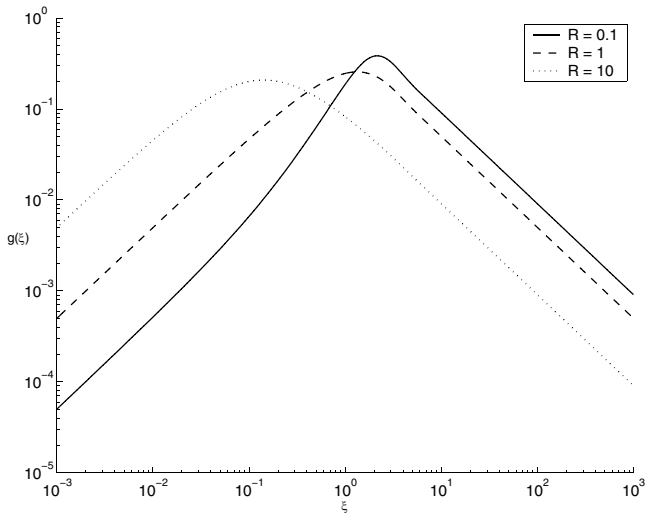
FIG. 4. Frequency-dependent part of thermoelastic loss function, $g(\xi)$ as a function of frequency, where $\xi \equiv \sqrt{2\omega\tau_f}$ and $\tau_f = l^2 C_f/k_f$. Curves are shown for three values of the parameter $R \equiv k_f\gamma_f/k_s\gamma_s = \sqrt{k_f C_f/k_s C_s}$. Note that peak height is only a weak function of R.

zero for $R = 1/\sqrt{3}$. At high frequencies, the dissipation falls off as $1/\sqrt{\omega}$.

The frequency dependence described by $g_m(\omega)$ for the contribution from the multilayer coating is simple, and essentially the same as that for conventional thermoelastic damping, so it needs little further discussion. It is important to note that for typical multilayer coatings, the characteristic time τ_m for the multilayer effects is much shorter than τ_f for the effects of the averaged uniform layer; for a coating with $2N$ layers,
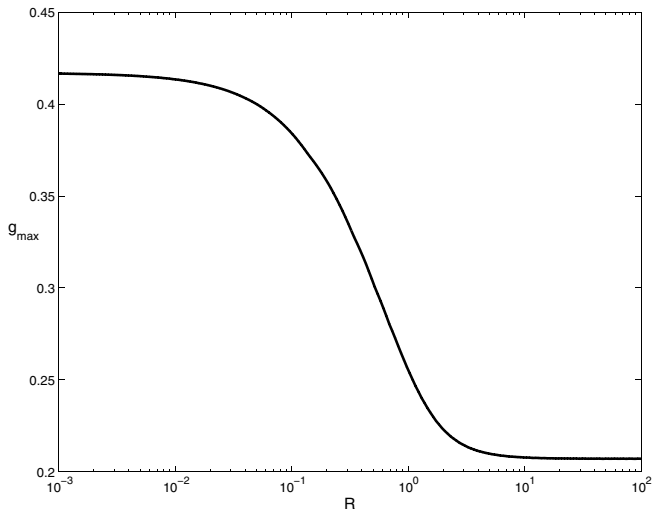
$$\tau_f/\tau_m = K_m^2 l^2 = 4\pi^2 N^2. \tag{62}$$

Since a typical high reflector that might be used in a LIGO interferometer has 40 layers, $\tau_f \sim 160\,00\tau_m$, so that the peak frequency for the contribution to the thermoelastic dissipation from thermal diffusion between the layers will be at a frequency $\sim$16 000 times higher, generally pushing the peak well above typical measurement ranges. The thermoelastic effects at frequencies of interest either for elastic Q measurements or thermal noise are thus generally dominated by the contributions of the averaged film.

D. Thermal noise

The results of Sec. III A indicate that the power dissipated by the thermoelastic effects can be comparable or even exceed that dissipated by the elastic loss in typical multilayer coatings. It is then reasonable to assume that the magnitude of the noise induced by the thermoelastic mechanism could be comparable to that from the elastic loss, and therefore must be calculated as part of the total noise budget for the coated mass.

Following the approach of [4], the displacement noise imposed on a Gaussian beam of normalized intensity distribution $I(r)$

$$I(r) = \frac{2}{\pi w^2} \exp\left(\frac{-2r^2}{w^2}\right) \tag{63}$$

is given by

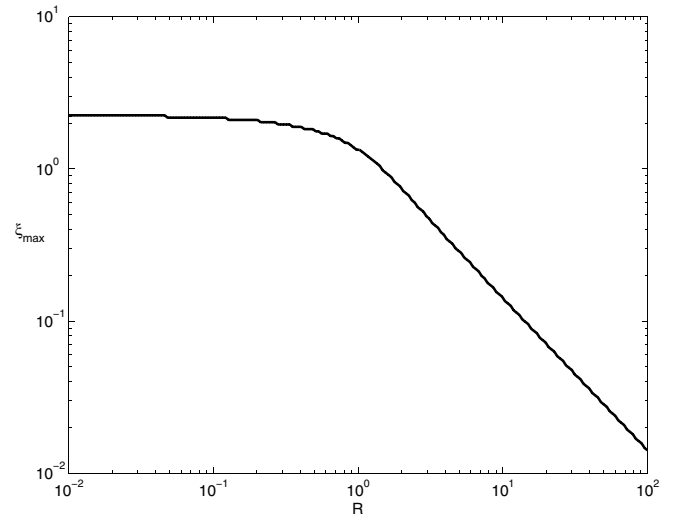$$S_x(f)df = \frac{2k_B T}{\pi^2 f^2} \frac{W_{\text{diss}}}{F_0^2} df \tag{64}$$

FIG. 5. Peak value of the normalized dissipation, $g_{\text{max}} \equiv g(\xi_{\text{max}})$ as a function of the parameter R. The peak value of the normalized dissipation is seen to be only a weak function of R, the only material parameter on which it depends.

FIG. 6. Normalized frequency ξ_{max} at which the normalized dissipation takes its maximum value, as a function of the parameter R. ξ_{max} corresponds to the thermal diffusion time across the film for $R < 1$, and decreases as $1/R$ (corresponding to ω_{max} decreasing as $1/R^2$ for $R > 1$).

where W_{diss} is the cycle-averaged power dissipated by a pressure field $\rho(r)$ oscillating at a frequency $\omega = 2\pi f$, of the same radial distribution as the intensity and with a resultant force F_0, i.e.

$$\rho(r) = F_0 I(r) \cos(\omega t). \tag{65}$$

Since the radius of the Gaussian beam is much larger than the thickness of the film or the thermal wave decay length, we can use the one-dimensional theory developed in previous Sections of this paper to evaluate W_{diss}. The zeroth-order elastic fields required for this calculation are available in [6]. We have from their Eqs. (A10)

$$
\begin{aligned}
\sigma_0(r) &\equiv \sigma_{0,zz} \\
&= -\rho(r) \\
\varepsilon_0(r) &\equiv [\varepsilon_{rr}(r) + \varepsilon_{\theta\theta}(r)]/2 \\
&= -\rho(r)/4(\lambda + \mu) \\
&= -\frac{(1 + \nu_s)(1 - 2\nu_s)}{2E_s} \rho(r)
\end{aligned}
\tag{66}
$$

where the third form of ε_0 follows from the definition of the Lamé constants in terms of the Young's modulus and Poisson ratio.

Here we analyze the noise due to an appropriately averaged uniform film, since the analysis of Sec. IV C 3 showed that the contribution to the dissipation associated with the thermal diffusion between the layers within the film are significant only at frequencies well above the LIGO detection band. We first consider a uniform film, and then the modifications necessary to describe an appropriately averaged multilayer.

Starting with Eq. (27) for the dissipated power per unit area, and Eqs. (20) for the thermal fields we have

$$
\begin{aligned}
\frac{P_{\text{diss}}}{\text{area}} &= \frac{\omega}{2} \{\alpha_f (D_f \varepsilon_0 + d_f \sigma_0)\text{Im}[-\theta_{1f}\gamma_f^{-1} \sinh(\gamma_f l)] \\
&\quad + \alpha_s (D_s \varepsilon_0 + d_s \sigma_0)\text{Im}[-\theta_{1s}\gamma_s^{-1} \exp(-\gamma_f l)]\} \\
&= \frac{\omega \Delta\beta l}{2}\left[\alpha_f (D_f \varepsilon_0 + d_f \sigma_0) \right. \\
&\quad \left. - \alpha_s (D_s \varepsilon_0 + d_s \sigma_0)\frac{C_f}{C_s} \right] g(\omega)
\end{aligned}
\tag{67}
$$

where as in Eqs. (34) and (35), we simplified the result by noting that only the homogeneous part of the thermal solutions contains an imaginary part, and in the second form used Eqs. (B6) and (B7) for $\theta_{1,f}$ and $\theta_{1,s}$. The frequency dependence is contained in the same function $g(\omega)$ defined in Eq. (38).

For this case, where there are both axial stresses and in-plane strains, we can calculate $\Delta\beta \equiv \beta_f - \beta_s$ from Eqs. (15), (16), (A1), (A8), and (A15). We find that

$$
\begin{aligned}
\beta_j &= \frac{E_j \alpha_j T}{C_j} \frac{1}{1 - \nu_j}\left[2\varepsilon_0 + \frac{1 + \nu_j}{E_j}\sigma_0 \right] \\
&= \frac{E_j \alpha_j T}{C_j} \frac{1}{1 - \nu_j}\left[\frac{(1 + \nu_s)(1 - 2\nu_s)}{E_s} + \frac{1 + \nu_j}{E_j} \right]\sigma_0
\end{aligned}
\tag{68}
$$

where the second follows for the specific form of the elastic fields given in Eqs. (66). With Eqs. (C8) and (C9) for the combinations of elastic constants represented by D_j and d_j, and $\Delta\beta$ calculated from Eq. (68), the result in Eq. (67) for the dissipated power per unit area becomes

$$
\begin{aligned}
\frac{P_{\text{diss}}(r)}{\text{area}} &= \frac{\omega T l C_f}{2}\rho(r)^2 g(\omega) \\
&\times \left\{ \frac{\alpha_f}{C_f}\left[\frac{1 + \nu_f}{1 - \nu_f} + \frac{(1 + \nu_s)(1 - 2\nu_s)}{1 - \nu_f}\frac{E_f}{E_s} \right] \right. \\
&\quad \left. - \frac{\alpha_s}{C_s}2(1 + \nu_s) \right\}^2
\end{aligned}
\tag{69}
$$

where we replaced σ_0^2 with $\rho(r)^2$ according to Eq. (66). Integrating over the infinite cross section to obtain the total dissipated power W_{diss}, and inserting that result into Eq. (64) for $S_x(f)$, we finally obtain

$$
\begin{aligned}
S_x(f) &= \frac{8k_B T^2}{\pi^2 f}\frac{l}{w^2}C_f g(\omega) \\
&\times \left\{ \frac{\alpha_f}{C_f}\frac{1}{2}\left[\frac{1 + \nu_f}{1 - \nu_f} + \frac{(1 + \nu_s)(1 - 2\nu_s)}{1 - \nu_f}\frac{E_f}{E_s} \right] \right. \\
&\quad \left. - \frac{\alpha_s}{C_s}(1 + \nu_s) \right\}^2 \\
&= \frac{8k_B T^2}{\pi^2 f}\frac{l}{w^2}\frac{\alpha_s^2 C_f}{C_s^2}(1 + \nu_s)^2 \Delta^2 g(\omega)
\end{aligned}
\tag{70}
$$

where Δ^2 is a dimensionless positive-definite combination of material constants that vanish when the film and substrate are identical,

$$\Delta^2 \equiv \left\{ \frac{C_s}{2\alpha_s C_f}\frac{\alpha_f}{(1 - \nu_f)}\left[\frac{1 + \nu_f}{1 + \nu_s} + (1 - 2\nu_s)\frac{E_f}{E_s} \right] - 1 \right\}^2. \tag{71}$$

Equation (70) is the final result for the thermoelastic displacement noise associated with a uniform coating. The frequency dependence represented by $g(\omega)$ is discussed at length in Sec. IV C 3.

An accurate approximation for a multilayer coating can be obtained by using a suitable averaging process to model it as a uniform layer, as discussed in Appendix D. Following the procedure described there, the result in Eq. (70) is replaced by

$$S_x(f) = \frac{8k_B T^2}{\pi^2 f}\frac{l}{w^2}\frac{\alpha_s^2 C_F}{C_s^2}(1 + \nu_s)^2 \tilde{\Delta}^2 g(\omega) \tag{72}$$

where

$$\tilde{\Delta}^2 \equiv \left\{ \frac{C_s}{2\alpha_s C_F} \left(\frac{\alpha}{1-\nu} \left[\frac{1+\nu}{1+\nu_s} + (1-2\nu_s)\frac{E}{E_s} \right] \right)_{\text{avg}} - 1 \right\}^2, \tag{73}$$

and the frequency dependence $g(\omega)$ is unchanged except for replacing the time constant τ_f by an appropriately averaged one τ_F. The volume-weighted average indicated by $(X)_{\text{avg}}$ is defined in Eq. (D1), the averaged heat capacity C_F in Eq. (D4), and τ_F in Eq. (D6). Since for room-temperature operation the thermoelastic noise is most important at frequencies falling in the low-frequency limit of $g(\omega)$, it is useful to insert into Eq. (72) the approximate result for g given in Eq. (60) to obtain

$$S_x(f) \rightarrow \frac{8\sqrt{2}k_B T^2}{\pi\sqrt{\omega}} \frac{l^2}{w^2} (1+\nu_s)^2 \frac{C_F^2}{C_s^2} \frac{\alpha_s^2}{\sqrt{k_s C_s}} \tilde{\Delta}^2. \tag{74}$$

Note that, as seen in Fig. 3, this low-frequency limit becomes inaccurate at the upper end of the gravitational wave detection band.

V. SUMMARY AND CONCLUSIONS

In this paper we have derived expressions for the thermoelastic dissipation associated with a coating on a test mass. For strains of the type consistent with mechanical loss measurements, numerical evaluation of the thermoelastic loss factors for coating/test-mass material combinations of the type being considered for use in future gravitational wave interferometers shows that thermoelastic dissipation is of a level comparable to that predicted to affect the sensitivity of advanced interferometers. Also derived is an expression for the expected power spectral density of thermoelastic noise from the coating of a mirror interrogated with a Gaussian beam. Evaluating this expression across the gravitational wave detector band using plausible values for the material parameters of coatings and substrates results in displacement noise that in some cases exceeds typical design sensitivities.

It should be noted that the expected thermoelastic noise is a strong function of the difference of the material parameters in the substrate and coating, so that the same coating will have different thermoelastic losses on different substrates. As many of the necessary material parameters are not well characterized, the noise levels calculated here should be considered as estimates only. Further experimental measurements of coating dissipation for likely choices for coating and substrate materials, and better characterization of the intrinsic coating thermophysical properties, should allow more accurate determination of the magnitude of the thermoelastic effects.

APPENDIX A: ZEROTH-ORDER DRIVING FIELDS

We need a set of zeroth-order driving fields that are consistent with the elastic boundary conditions. In all cases we must have continuity of the in-plane strains and the normal stress at the film-substrate interface, $z = l$: $\varepsilon_{0,xx,s} = \varepsilon_{0,xx,f}$, $\varepsilon_{0,yy,s} = \varepsilon_{0,yy,f}$, and $\sigma_{0,zz,f} = \sigma_{0,zz,s}$. As discussed in Sec. I, the pertinent elastic fields can be specified in terms of two components, the in-plane dilation $\varepsilon_0 \equiv (\varepsilon_{0,xx} + \varepsilon_{0,yy})/2$ and the axial stress $\sigma_0 \equiv \sigma_{0,zz}$, which are independent of z under the assumptions set up in Sec. I. We can neglect the antisymmetric in-plane strain ($\varepsilon_{0,xx} - \varepsilon_{0,yy}$) which does not interact thermoelastically (as is shown in Sec. IV C 1), and, for convenience, can take $\varepsilon_{0,xx} = \varepsilon_{0,yy} = \varepsilon_0$.

We consider two cases, a stress-free surface with a specified in-plane strain ($\sigma_0 = 0$ and ε_0 specified), and a specified surface-normal stress with a vanishing in-plane strain ($\varepsilon_0 = 0$ and σ_0 specified). Any elastic state pertinent to the thermoelastic problem can be obtained as an appropriately weighted sum of these two solutions. For the general case, where both σ_0 and ε_0 are nonzero, one of the important results of this appendix, Σ_j defined in Eq. (16), can be written

$$\Sigma_j \equiv \sum_{i=1}^{3} \varepsilon_{0,ii,j} \quad = S_j \varepsilon_0 + s_j \sigma_0. \tag{A1}$$

The combination of elastic constants S_j and s_j are obtained in this Appendix, Eqs. (A8) and (A15), respectively.

1. Specified in-plane strain, stress-free surface

For a stress-free surface of the mass, as would be the case for a Q measurement, we have $\sigma_0 = 0$ and ε_0 specified. Noting that the under these assumptions the continuity condition on the normal stress implies that $\sigma_{0,zz,s} = \sigma_{0,zz,f} = \sigma_0 = 0$, the only unknown field components are $\varepsilon_{0,zz,f}$, $\varepsilon_{0,zz,s}$, $\sigma_{0,xx,f} = \sigma_{0,yy,f} \equiv \sigma_{0,\parallel,f}$, and $\sigma_{0,\parallel,s}$. The symmetry of the problem allowed us to take $\sigma_{0,xx} = \sigma_{0,yy} \equiv \sigma_{0,\parallel}$.

Begin with Hooke's law, Eq. 5.14 of [10]:

$$\varepsilon_{0,zz,j} = \frac{1}{E_j}[\sigma_{0,zz,j} - \nu_j(\sigma_{0,xx,j} + \sigma_{0,yy,j})]$$

$$= \frac{-2\nu_j}{E_j}\sigma_{0,\|,j}. \tag{A2}$$

where we recall the notation that a subscript $j = f, s$ stands for a quantity evaluated in the film or substrate, respectively. Summing the expressions for the in-plane strains in [10], it follows that

$$\varepsilon_{0,xx,j} + \varepsilon_{0,yy,j} = \frac{1}{E_j}[\sigma_{0,xx,j} + \sigma_{0,yy,j} - \nu_j(\sigma_{0,xx,j} + \sigma_{0,yy,j})] \tag{A3}$$

or equivalently

$$\varepsilon_0 = \frac{1 - \nu_j}{E_j}\sigma_{0,\|,j}. \tag{A4}$$

Finally, going back to Eq. (A2) with Eq. (A4), we find

$$\varepsilon_{0,zz,j} = \frac{-2\nu_j}{1 - \nu_j}\varepsilon_0. \tag{A5}$$

It is convenient to summarize these results for the zeroth-order elastic fields in the form given in Eq. (23),

$$\varepsilon_{0,ii,j} = A_{0,ii,j}\varepsilon_0, \qquad \sigma_{0,ii,j} = B_{0,ii,j}\varepsilon_0, \tag{A6}$$

where

$$A_{0,xx,j} = A_{0,yy,j} = 1,$$
$$A_{0,zz,j} = \frac{-2\nu_j}{1 - \nu_j}$$
$$B_{0,xx,j} = B_{0,yy,j} = \frac{E_j}{1 - \nu_j},$$
$$B_{0,zz,j} = 0 \tag{A7}$$

A result, used in Eq. (13), is the evaluation of a sum over strains introduced in Eq. (16), which, with Eqs. (A1) and (A7), becomes

$$S_j\varepsilon_0 \equiv \sum_{i=1}^{3}\varepsilon_{0,ii,j} = \sum_{i=1}^{3}A_{0,ii,j}\varepsilon_0 = \frac{2(1 - 2\nu_j)}{1 - \nu_j}\varepsilon_0. \tag{A8}$$

Another result, used in Eq. (29) to evaluate the energy stored in the film is

$$U_f \equiv \sum_{i=1}^{3}B_{0,ii,f}A_{0,ii,f} = \frac{2E_f}{1 - \nu_f}. \tag{A9}$$

2. Specified surface-normal stress, vanishing in-plane strain

For a specified surface-normal stress, as would be the case for calculating thermal noise, we have $\sigma_{0,zz,f} = \sigma_0$.

To make this case complementary to that in Appendix A 1, we assume vanishing in-plane strains, i.e. $\varepsilon_{0,xx} = \varepsilon_{0,yy} \equiv \epsilon_0 = 0$ in both the film and the substrate. Noting that the under these assumptions the continuity condition on the normal stress implies that $\sigma_{0,zz,s} = \sigma_{0,zz,f} = \sigma_0$, so the only unknown field components are $\varepsilon_{0,zz,j}$ and $\sigma_{0,xx,j} = \sigma_{0,yy,j} \equiv \sigma_{0,\|,j}$, for $j = f, s$. The analysis is similar to that in Appendix A 1. Begin with Eq. 5.14 of [10]:

$$\varepsilon_{0,xx,j} = \frac{1}{E_j}[\sigma_{0,xx,j} - \nu_j(\sigma_{0,yy,j} + \sigma_{0,zz,j})] \tag{A10}$$

which can be solved with $\sigma_{0,xx,j} = \sigma_{0,yy,j} \equiv \sigma_{0,\|,j}$ to yield

$$\sigma_{0,\|,j} = \frac{\nu_j}{1 - \nu_j}\sigma_0. \tag{A11}$$

With another of Eqs. 5.14 from [10]:

$$\varepsilon_{0,zz,j} = \frac{1}{E_j}[\sigma_{0,zz,j} - \nu_j(\sigma_{0,xx,j} + \sigma_{0,yy,j})] \tag{A12}$$

and Eq. (A11) we obtain

$$\varepsilon_{0,zz,j} = \frac{\sigma_0}{E_j}\frac{(1 - 2\nu_j)(1 + \nu_j)}{1 - \nu_j}. \tag{A13}$$

We can again collect the results of this section in the form given in Eqs. (23),

$$\varepsilon_{0,ii,j} = a_{0,ii,j}\sigma_0,$$
$$\sigma_{0,ii,j} = b_{0,ii,j}\sigma_0,$$
$$a_{0,xx,j} = a_{0,yy,j} = 0,$$
$$a_{0,zz,j} = \frac{(1 - 2\nu_j)(1 + \nu_j)}{1 - \nu_j}\frac{1}{E_j}$$
$$b_{0,xx,j} = b_{0,yy,j} = \frac{\nu_j}{1 - \nu_j},$$
$$b_{0,zz,j} = 1. \tag{A14}$$

A result, used in Eq. (13), is the evaluation of a sum over strains introduced in Eq. (16), which, with Eqs. (A1) and (A14) can be written

$$s_j\sigma_0 \equiv \sum_{i=1}^{3}\varepsilon_{0,ii,j} = \sum_{i=1}^{3}a_{0,ii,j}\sigma_0$$
$$= \frac{(1 - 2\nu_j)(1 + \nu_j)}{1 - \nu_j}\frac{1}{E_j}\sigma_0. \tag{A15}$$

Another result, used in Eq. (40) to evaluate the energy stored in the film is

$$u_f \equiv \sum_{i=1}^{3}b_{0,ii,f}a_{0,ii,f} = \frac{(1 - 2\nu_f)(1 + \nu_f)}{E_f(1 - \nu_f)}. \tag{A16}$$

APPENDIX B: THE THERMAL FIELDS FOR TWO IMPORTANT CASES

The unknown coefficients in the homogeneous parts of the thermal fields, Eqs. (18), can be obtained from the particular solutions, Eqs. (20), and the boundary conditions, Eqs. (17). Continuity of the thermal field at $z = l$ requires

$$\theta_{p,f}(l) + \theta_{1f}\cosh(\gamma_f l) = \theta_{p,s} + \theta_{1s}e^{-\gamma_s l} \qquad (B1)$$

while continuity of the thermal flux requires

$$k_f[\theta'_{p,f}(l) + \theta_{1f}\gamma_f\sinh(\gamma_f l)] = -k_s\theta_{1s}\gamma_s e^{-\gamma_s l}. \quad (B2)$$

Simultaneous solution of these equations yields

$$\theta_{1,f} = \frac{[\theta_{p,s} - \theta_{p,f}(l)] - (R/\gamma_f)\theta'_{p,f}(l)}{\cosh(\gamma_f l) + R\sinh(\gamma_f l)} \qquad (B3)$$

and

$$\theta_{1,s} =$$
$$-e^{\gamma_s l}\frac{[\theta_{p,s} - \theta_{p,f}(l)]R\sinh(\gamma_f l) + (R/\gamma_f)\cosh(\gamma_f l)\theta'_{p,f}(l)}{\cosh(\gamma_f l) + R\sinh(\gamma_f l)}.$$
$$(B4)$$

where $R \equiv k_f\gamma_f/k_s\gamma_s$. To make further progress, it is necessary to find the particular solutions for specific cases. We consider here two cases of interest, a uniform film on a uniform substrate, and a periodic film on a uniform substrate.

1. Uniform film on uniform substrate

Consider first both the film and substrate to be uniform. By inspection of the thermal field equation, Eq. (14), particular solutions for this case are constant and given by

$$\theta_{p,j}(z) = -\beta_j. \qquad (B5)$$

With Eq. (B5) for the particular solutions, the coefficients in the homogeneous solutions in film and substrate from Eqs. (B3) and (B4) become

$$\theta_{1,f} = \frac{\Delta\beta}{\cosh(\gamma_f l) + R\sinh(\gamma_f l)} \qquad (B6)$$

and

$$\theta_{1,s} = -e^{\gamma_s l}\frac{\Delta\beta R\sinh(\gamma_f l)}{\cosh(\gamma_f l) + R\sinh(\gamma_f l)}. \qquad (B7)$$

where $\Delta\beta \equiv \beta_f - \beta_s$.

2. Modulated film on uniform substrate

In the case of a nonuniform film, the expression for the particular solution is somewhat more complicated. Assume a film whose thermal conductivity takes the form

$$\alpha_f(z) = \bar{\alpha}_f + \alpha_m\cos(K_m z), \qquad (B8)$$

in which case with Eq. (15) $\beta_f(z)$ takes the form

$$\beta_f(z) = \beta_f + \beta_m\cos(K_m z) \qquad (B9)$$

where

$$\beta_f \equiv \frac{E_f\bar{\alpha}_f T}{C_f}\frac{\Sigma_f}{1 - 2\nu_f} \quad \text{and} \quad \beta_m \equiv \frac{E_f\alpha_m T}{C_f}\frac{\Sigma_f}{1 - 2\nu_f}. \qquad (B10)$$

With Eq. (B9), the thermal field equation Eq. (14) takes the form

$$\frac{\partial^2\theta_{p,f}(z)}{\partial z^2} = \frac{i\omega}{\kappa_f}[\theta_{p,f}(z) + \beta_f + \beta_m\cos(K_m z)]$$
$$= \gamma_f^2[\theta_{p,f}(z) + \beta_f + \beta_m\cos(K_m z)] \qquad (B11)$$

where the definition $\gamma_f^2 = i\omega/\kappa_f$ from Eq. (19) was used to obtain the second form. The particular solution has two terms, a constant part similar to that in Eq. (B5) for the uniform case, and one that has a spatial variation that follows the thermal expansion coefficient. To obtain the periodic part, take an ansatz $\theta_{p,f}(z) = q\cos(K_m z)$. With this ansatz in Eq. (B11) we obtain

$$(\gamma_f^2 + K_m^2)q = -\gamma_f^2\beta_m \qquad (B12)$$

Combining with the constant part we obtain the total particular solution in the film,

$$\theta_{p,f}(z) = -\beta_f - \beta_m\frac{\gamma_f^2}{\gamma_f^2 + K_m^2}\cos(K_m z). \qquad (B13)$$

For the assumed uniform substrate, the particular solution is like that in Eq. (B5), i.e.

$$\theta_{p,s} = -\beta_s. \qquad (B14)$$

With Eqs. (B13) and (14) for the particular solutions, the coefficients in the homogeneous solutions in film and substrate from Eqs. (B3) and (B4) become

$$\theta_{1,f} = \frac{\Delta\beta}{\cosh(\gamma_f l) + R\sinh(\gamma_f l)} + \beta_m\frac{\gamma_f^2}{\gamma_f^2 + K_m^2}$$
$$\times\frac{\cos(K_m l) - (RK_m/\gamma_f)\sin(K_m l)}{\cosh(\gamma_f l) + R\sinh(\gamma_f l)} \qquad (B15)$$

and

$$\theta_{1,s} = -\Delta\beta e^{\gamma_s l}\frac{R\sinh(\gamma_f l)}{\cosh(\gamma_f l) + R\sinh(\gamma_f l)}$$
$$-\beta_m R e^{\gamma_s l}\frac{\gamma_f^2}{\gamma_f^2 + K_m^2}$$
$$\times\frac{\cos(K_m l)\sinh(\gamma_f l) - (K_m/\gamma_f)\sin(K_m l)\cosh(\gamma_f l)}{\cosh(\gamma_f l) + R\sinh(\gamma_f l)}.$$
$$\text{(B16)}$$

where $\Delta\beta \equiv \beta_f - \beta_s$.

APPENDIX C. SOLVING FOR THE THERMOELASTICALLY GENERATED ELASTIC FIELDS

Given the solution Eqs. (20) for the oscillatory thermal field, we must solve for the thermally driven elastic fields, $\sigma_1(z)$ and $\varepsilon_1(z)$, whose imaginary parts lead to the dissipation in which we are interested. The boundary conditions are $\sigma_{1,zz} = 0$ at the stress-free surface $z = 0$, $\varepsilon_{1,xx} = \varepsilon_{1,yy} = 0$ for $z \to \infty$, and continuity of the in-plane strains $\varepsilon_{1,xx}$ and $\varepsilon_{1,yy}$, and the normal stress $\sigma_{1,zz}$, at the boundary between the film and substrate.

The point of departure is the equation of elastic equilibrium, Eq. 7.8 of [10],

$$\frac{d}{dz}[\varepsilon_{xx} + \varepsilon_{yy} + 2(1 - \nu)\varepsilon_{zz} - 2(1 + \nu)\alpha\theta] = 0 \quad \text{(C1)}$$

adapted here by dividing Landau's α by three to convert it from volumetric to linear expansion, replacing $\alpha\nabla\theta$ with $\nabla(\alpha\theta)$ to accommodate a possible spatial variation in the thermal expansion coefficient, and specializing to stress and strain fields that depend only on z. Hooke's law in the presence of a nonuniform temperature field $\theta(z)$, Eq. 6.2 of [10] is:

$$\sigma_{zz} = \frac{E}{1 + \nu}\left[\varepsilon_{zz} + \frac{\nu}{1 - 2\nu}(\varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz})\right] - \frac{E\alpha\theta}{1 - 2\nu}.$$
$$\text{(C2)}$$

With the boundary condition $\sigma_{1,zz} = 0$, Eq. (C2) results in

$$\varepsilon_{1,zz,j} = -\frac{\nu_j}{1 - \nu_j}(\varepsilon_{1,xx,j} + \varepsilon_{1,yy,j}) + \frac{1 + \nu_j}{1 - \nu_j}\alpha_j\theta_j. \quad \text{(C3)}$$

Inserting Eq. (C3) into Eq. (C1), we find

$$\frac{d}{dz}[(1 - 2\nu_j)(\varepsilon_{1,xx,j} + \varepsilon_{1,yy,j})] = 0.$$

Noting the continuity of the in-plane strains, and their vanishing at infinity, we conclude $(\varepsilon_{1,xx,j} + \varepsilon_{1,yy,j}) = 0$. With this result, Eq. (C3) becomes

$$\varepsilon_{1,zz,j} = \frac{1 + \nu_j}{1 - \nu_j}\alpha_j\theta_j. \quad \text{(C4)}$$

With the Hooke's law expression for σ_{xx} analogous to Eq. (C2) for σ_{zz}, and inserting Eq. (C4), we obtain

$$\sigma_{1,xx,j} = -\frac{E_j\alpha_j\theta_j}{1 - \nu_j} \quad \text{(C5)}$$

and by symmetry $\sigma_{1,yy,j} = \sigma_{1,xx,j}$.

These results constitute a consistent set of first-order elastic fields. It is convenient to summarize them in the form:

$$\varepsilon_{1,ii,j}(z) = A_{1,ii,j}\alpha_j\theta_j, \qquad \sigma_{1,ii,j}(z) = B_{1,ii,j}\alpha_j\theta_j, \quad \text{(C6)}$$

where

$$A_{1,xx,j} = A_{1,yy,j} = 0, \quad A_{1,zz,j} = \frac{1 + \nu_j}{1 - \nu_j}$$

$$B_{1,xx,j} = B_{1,yy,j} = -\frac{E_j}{1 - \nu_j}, \quad B_{1,zz,j} = 0. \quad \text{(C7)}$$

Combinations of these parameters used in calculating the dissipated power, D_j and d_j in Eqs. (26), can be evaluated with Eqs. (A7) and (C7) as

$$D_j \equiv \sum_{i=1}^{3}[B_{0,ii,j}A_{1,ii,j} - B_{1,ii,j}A_{0,ii,j}] = \frac{2E_j}{1 - \nu_j}, \quad \text{(C8)}$$

and with Eqs. (A14) and (C7) as

$$d_j \equiv \sum_{i=1}^{3}[b_{0,ii,j}A_{1,ii,j} - B_{1,ii,j}a_{0,ii,j}] = \frac{1 + \nu_j}{1 - \nu_j}. \quad \text{(C9)}$$

APPENDIX D: AVERAGING MATERIAL PROPERTIES IN A PERIODIC MULTILAYER

In cases of practical interest, the optical coating is a multilayer rather than a homogeneous film. The analysis in Sections IV C 2 and IV C 3 indicated that for realistic cases either for Q measurements or for thermoelastic noise, the thermal diffusion length is large compared to the period of the multilayer. Hence, an analysis that treats the film as an effective homogeneous medium with suitably averaged properties should yield a result of adequate accuracy. It is then necessary to form the appropriate average of the various material properties involved. For simplicity, we take the coating to consist of alternating layers of two types of material, labeled a and b, of thicknesses d_a and d_b, respectively. We define the volume averaging operator by

$$(X)_{\text{avg}} \equiv \frac{d_a}{d_a + d_b}X_a + \frac{d_b}{d_a + d_b}X_b. \quad \text{(D1)}$$

1. Thermal field averaging

Consider first the heat equation. Define an average temperature in the film, $\theta_F(z)$, where we use the subscript F to indicate a quantity in the film suitably averaged over a period of the structure (averaging will be different for different quantities), under the assumption that the dis-

tance over which this averaged quantity varies significantly is much greater than the period $d_a + d_b$. Since the temperature field is continuous at the boundaries between the layers, $\theta_F(z) = \theta_a(z) = \theta_b(z)$. To obtain an averaged heat equation for the propagation of θ_F, begin with Eq. (14), here rewritten in a more convenient form,

$$i\omega C_q \theta_q(z) - \frac{\partial}{\partial z}\left(k_q \frac{\partial \theta_q}{\partial z}\right) = -i\omega C_q \beta_q, \qquad (D2)$$

where $q = a, b$ indicates a quantity evaluated in layer a or b, respectively. Averaging the first and last terms over a period of the structure is trivial. The second term requires more care. Noting that the continuity of the heat flux requires that $k_a \partial \theta_a / \partial z = k_b \partial \theta_b / \partial z = (k \partial \theta / \partial z)_{\text{avg}}$, implicitly defining the averaged thermal conductivity k_F by writing the average heat flux as

$$k_F\left(\frac{\partial \theta_F}{\partial z}\right)_{\text{avg}} \equiv \left(k\frac{\partial \theta}{\partial z}\right)_{\text{avg}},$$

and solving for k_F, we find

$$k_F^{-1} = (k^{-1})_{\text{avg}}. \qquad (D3)$$

We can then write the averaged Eq. (D2) in a form analogous to Eq. (14),

$$i\omega \theta_F(z) - \kappa_F \frac{\partial^2 \theta_F}{\partial z^2} = -i\omega \beta_F, \qquad (D4)$$

where the averaged film properties are

$$C_F \equiv (C)_{\text{avg}} \qquad \kappa_F \equiv k_F/C_F$$

$$\beta_F \equiv \frac{(C\beta)_{\text{avg}}}{C_F} = \frac{1}{C_F}\left(\frac{E\alpha T\Sigma}{1 - 2\nu}\right)_{\text{avg}} \qquad (D5)$$

where we used Eq. (15) for β. With these averaged quantities in place of those of the uniform film, i.e., taking $X_f \rightarrow X_F$, we can immediately transcribe all the previous results for the temperature field in the uniform film without further analysis. It is also convenient to define a thermal diffusion time for the averaged film of thickness l,

$$\tau_F \equiv l^2/\kappa_F. \qquad (D6)$$

2. Elastic fields in a multilayer

Averaging of the elastic properties is more straightforward. The zeroth-order elastic fields already taken as invariant through the region of interest, i.e., the in-plane dilation $\varepsilon_0 \equiv (\varepsilon_{0,xx} + \varepsilon_{0,yy})/2$ and the axial stress $\sigma_0 \equiv \sigma_{0,zz}$, remain invariant in the multilayer, so they are obviously equal to their average.

The calculation of the remaining components of the elastic field then follows exactly as given in Appendix A, so that the correct result for the fields in material $q = a, b$ in a modulated film can be obtained from the correspond-

ing expression for the zeroth-order field in a uniform film $j = f$ by replacing $j \rightarrow q$. For example, for the case ε_0 specified and $\sigma_0 = 0$ we simply have, analogously to Eq. (A6)

$$\varepsilon_{0,ii,q} = A_{0,ii,q}\varepsilon_0, \qquad \sigma_{0,ii,q} = B_{0,ii,q}\varepsilon_0, \qquad (D7)$$

where

$$A_{0,xx,q} = A_{0,yy,q} = 1, \qquad A_{0,zz,q} = \frac{-2\nu_q}{1 - \nu_q}$$

$$B_{0,xx,q} = B_{0,yy,q} = \frac{E_q}{1 - \nu_q}, \qquad B_{0,zz,q} = 0. \qquad (D8)$$

The same approach provides the results for $a_{0,ii,q}, b_{0,ii,q}, \Sigma_{0,ii,q}, S_q, s_q$, from the corresponding expressions in Eqs. (A14), (A1), (A8), and (A15), respectively.

For the power stored in the film, analogous to Eq. (A9), we must average the energy stored in the components of the multilayer,

$$U_F = \left(\frac{2E}{1 - \nu}\right)_{\text{avg}}. \qquad (D9)$$

By similar arguments as were applied to the zeroth-order fields, the first-order fields analogous to those obtained for a uniform film in Appendix C, can be obtained from the corresponding expression for the first-order field in a uniform film $j = f$ by replacing $j \rightarrow q$. For example, analogously to Eqs. (C6),

$$\varepsilon_{1,ii,q}(z) = A_{1,ii,q}\alpha_q \theta_F(z), \qquad \sigma_{1,ii,q}(z) = B_{1,ii,q}\alpha_q \theta_F(z), \qquad (D10)$$

where

$$A_{1,xx,q} = A_{1,yy,q} = 0, \qquad A_{1,zz,q} = \frac{1 + \nu_q}{1 - \nu_q}$$

$$B_{1,xx,q} = B_{1,yy,q} = -\frac{E_q}{1 - \nu_q}, \qquad B_{1,zz,q} = 0. \qquad (D11)$$

The combinations of these parameters used in calculating the dissipated power, D_q and d_q, analogous to Eqs. (C8) and (C9), are obtained similarly.

3. Averaging the dissipated power

To find the averaged dissipated power, start with Eq. (27). Noting that the temperature is continuous and slowly varying over a period of the structure, we can write

$$\frac{P_{\text{diss}}}{\text{area}} = \frac{\omega}{2}\left\{[(D\varepsilon_0 + d\sigma_0)\alpha]_{\text{avg}} \int_0^l \text{Im}[-\theta_F(z)]dz \right.$$
$$\left. + (D_s\varepsilon_0 + d_s\sigma_0)\int_l^\infty \text{Im}[-\theta_s(z)]dz\right\}.$$

Note that in making this approximation, we exclude cases where the thermal diffusion length approaches the layer

period, but do allow the thermal length to be less than to the total thickness of the multilayer. This is not a very restrictive assumption: for typical mirror films of ~ 20 layer pairs, frequencies up to $\sim 10^4$ above the dissipation peak are allowed [see Eq. (62)]. Comparing with Eq. (27), we see that any result for a uniform film can be transformed into the corresponding result for the averaged film by replacing

$$(D_f \varepsilon_0 + d_f \sigma_0) \alpha_f \rightarrow [(D \varepsilon_0 + d \sigma_0) \alpha]_{\text{avg}}, \qquad \text{(D13)}$$

where the elastic quantities required are given in Eqs. (C8) and (C9), and the averaging operation is defined in Eq. (D1); we again replace the thermal properties with the averaged ones given in Eqs. (D3) and (D5), i.e. $X_f \rightarrow X_F$. Since the dissipated power is the key quantity from which all the end results of this paper devolve, only straightforward substitution and algebraic manipulation are required to obtain those results for the averaged film. The results so obtained for the dissipation factor ϕ and spectral density of thermoelastic noise $S_x(f)$ are given in Eqs. (42) and (72), respectively.

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