Estimating the off resonance thermal noise in mirrors, Fabry-Perot interferometers, and delay lines: The half infinite mirror with uniform loss

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We present an analysis comparing the thermal noise in Fabry-Perot and delay-line interferometers with half-infinite mirrors. With a center to center spot spacing of at least twice the Gaussian beam spot size a delay line produces significantly less phase noise than a comparable Fabry-Perot interferometer for the case of half infinite mirrors at frequencies where test mass thermal noise usually dominates. When the delay-line spots overlap substantially the delay-line produces approximately the same noise as the Fabry-Perot interferometer. For a single bounce these results agree with those based on the method of Levin.

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I. INTRODUCTION

Several large interferometric gravitational wave detectors will begin regular astronomical observations over the next few years; Laser Interferometric Gravitational Wave Observatory LIGO (USA) [1], VIRGO (French-Italian) [2], and GEO 600 (British-German) [3]. The fundamental sources of noise in the first-generation LIGO interferometer will be seismic noise filtered through the mirror suspension at low frequency ($f<50$ Hz), shot noise at high frequency ($f>200$ Hz), and thermal noise in the fused silica interferometer mirrors at intermediate frequencies ($50\text{ Hz}<f<200$ Hz). Gillespie and Raab calculated this last noise source using a normal-mode expansion of the displacements in the mirrors [4]. More direct approaches to the problem have been developed by Levin [5], Nakagawa *et al.* [6], and Bondu, Hello, and Vinet [7]. These nonmodal approaches often lead to simpler expressions for model problems, offering greater insight into the parameters determining the magnitude of the thermal noise. They also serve as the basis for more efficient computational methods than the modal methods.

The analysis in our previous paper [6] began with the general fluctuation dissipation theorem

$$\langle u_i(\vec{r}_1)u_j(\vec{r}_2)\rangle_\omega=\frac{2k_BT}{\omega}\,\mathrm{Im}\,\chi^{\omega}_{ij}(\vec{r}_1,\vec{r}_2;\mathbf{c}), \tag{1}$$

which relates the cross-spectral density of the displacements u_i of a body to the elastic response function χ^{ω}_{ij} of that body, where k_B is the Boltzmann constant, T is the temperature, $\vec{r}$ is a surface point, and $\mathbf{c}=c_{ijkl}$ is the generalized stiffness tensor, which can be divided into a real and an imaginary part:

$$c_{ijkl}(\omega)=c'_{ijkl}(\omega)-ic''_{ijkl}(\omega). \tag{2}$$

Specifically, the response function $\chi^{\omega}_{ij}(\vec{x},\vec{r})$ is a dynamic Green's function at frequency ω satisfying the traction-free boundary condition, relating the displacement u_i at volume point $\vec{x}$ resulting from $\mathcal{T}_j(\vec{r})=\mathcal{T}_{jk}(\vec{r})n_k$, where $\mathcal{T}_{ij}$ is a stress tensor at the surface point $\vec{r}$, and $\vec{n}$ is the unit vector normal to the surface. It is shown in Ref. [6] that in the quasistatic approximation the fluctuation dissipation theorem can be written

$$\langle u_i(\vec{r}_1)u_j(\vec{r}_2)\rangle_\omega\approx\frac{2k_BT}{\omega}\int_V dV[\partial_k\chi^{\omega}_{li}(\vec{x},\vec{r}_1;\mathbf{c}')]c''_{klpq}(\omega)\times[\partial_p\chi^{\omega}_{qj}(\vec{x},\vec{r}_2;\mathbf{c}')], \tag{3}$$

where we replaced the full Green's function with the lossless Green's function and hence eliminate the imaginary part of the stiffness tensor from the argument of the Green's function. Moreover, when the loss function is the same for all components of the stiffness tensor and we consider the quasistatic limit, we can rewrite Eq. (3) as

$$\langle u_i(\vec{r}_1)u_j(\vec{r}_2)\rangle_\omega=\frac{k_BT}{\omega}\,\phi(\omega)\{\chi^{\text{static}}_{ij}(\vec{r}_1,\vec{r}_2;\mathbf{c}')+\chi^{\text{static}}_{ji}(\vec{r}_2,\vec{r}_1;\mathbf{c}')\}. \tag{4}$$

The loss function of the material, $\phi(\omega)$, can be factored in this fashion only for the special case where, for each component of the stiffness tensor, the real and imaginary parts are in the same ratio so that $c''_{ijkl}(\omega)=\phi(\omega)c'_{ijkl}(\omega)$. We derive Eq. (4) from Eq. (3) after integration by parts, using the traction-free boundary condition, the field equation, and the uniformity of the loss. We see that Eq. (4) is valid only when all the above assumptions are satisfied, including the single loss function assumption. In more general cases such as localized lossy layers, Eq. (4) will have a modified form, still derivable from the general quasistatic formula, Eq. (3).

In the initial large interferometric gravitational wave detectors the lowest-frequency normal mode of the test masses will be above 6 kHz, while the highest frequency at which the test mass thermal noise is larger than the shot noise will be approximately 200 Hz. Thus, the quasistatic approximation will be valid over the range of interest for practical applications. Here, the Green's function approach is particularly advantageous compared to the modal approach, which requires summing over many highly detuned modes. In this paper we discuss half-infinite mirrors, for which the quasi-static approximation is better described by comparing the time required for elastic waves in the test mass to cross the laser beam spot to the period of the highest-frequency gravitational wave for which test mass thermal noise is of interest.

In this paper we derive expressions relating the cross-spectral density of the mirror displacements to the phase noise imposed on a Gaussian light beam reflected from a mirror, on a beam reflected from a Fabry-Perot interferometer, and on a beam passing through a delay line. We then evaluate these three expressions for the spectral density of phase noise treating the mirrors as half-infinite in extent and with uniform loss. For the case of a single bounce from a mirror we obtain Levin's [5] result as corrected by Liu and Thorne [8].

II. PHASE NOISE OF A GAUSSIAN BEAM FROM A SINGLE MIRROR REFLECTION

A mirror at finite temperature will have a surface which is distorted by Brownian motion. The electric field at a plane z and time t associated with an incident TEM_{00} Gaussian light beam of $1/e$ field radius w reflected from a mirror with surface distortions $\delta z(x,y,t)$ can be written, in the near field, as a Gaussian beam with a small phase shift that is a function of the position on the mirror $\vec{r}=(x,y)$ and the time t,

$$E_{\mathrm{ref}}(x,y,z,t)=E_0 e^{-|\vec{r}|^2/w^2} e^{i(kz-\omega_0 t)} e^{2ik\delta z(x,y,t)}. \tag{5}$$

This reflected field just after the mirror can be decomposed into a linear superposition of Hermite Gaussians,

$$E_{\mathrm{ref}}(x,y,z,t)=E_0 e^{i(kz-\omega_0 t)}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} A_{n,m} H_n(\sqrt{2}x/w) \times e^{-(x^2/w^2)} H_m(\sqrt{2}y/w) e^{-(y^2/w^2)}. \tag{6}$$

We need only retain the term in this expansion that corresponds to the TEM_{00} mode $(n=m=0)$ because, in the case of a gravitational wave detector with Fabry-Perot arms, only this mode will be resonant, while for a delay-line system a spatial filter at the output of the interferometer will select only this spatial mode. Equating Eqs. (5) and (6), and approximating the complex exponential in Eq. (5) to first order in $\delta z(x,y,t)$, the integral projecting out the coefficient of the TEM_{00} mode is

$$A_{0,0}(t)\approx 1+\frac{4ki}{\pi w^2}\int_{-\infty}^{\infty}dx\int_{-\infty}^{\infty}dy\, e^{-2|\vec{r}|^2/w^2}\delta z(x,y,t), \tag{7}$$

which means that the phase shift after reflection from a mirror is

$$\varphi(t)\approx \frac{4k}{\pi w^2}\int\int dS\, e^{-2|\vec{r}|^2/w^2}\delta z(\vec{r},t), \tag{8}$$

where $\int\int dS=\int\int dx\,dy$.

The autocorrelation function of the phase fluctuations generated by the random variable δz is

$$R_\varphi(t)=\langle \varphi(t)\varphi(0)\rangle, \tag{9}$$

where the notation $\langle\ \rangle$ denotes the statistical expectation operator, which can be evaluated using $\varphi(t)$ from Eq. (8) by interchanging the order of the statistical averaging and spatial integrations to obtain

$$R_\varphi(t)=\frac{16}{\pi^2}\frac{k^2}{w^4}\int\int dS\int\int dS'\, e^{-2|\vec{r}|^2/w^2} e^{-2|\vec{r}\,'|^2/w^2} \times\langle \delta z(\vec{r},t)\,\delta z(\vec{r}\,',0)\rangle. \tag{10}$$

The Fourier transform of the autocorrelation function of the phase fluctuations induced by a single bounce from a mirror is the power spectral density of the phase noise,

$$S_\varphi^{\mathrm{single}}(\omega)=\int_{-\infty}^{\infty}dt\, e^{i\omega t}R_\varphi(t)=\int_{-\infty}^{\infty}dt\, e^{i\omega t}\langle \varphi(t)\varphi(0)\rangle. \tag{11}$$

Inserting Eq. (8) into Eq. (11) and noting that the Fourier transform of the autocorrelation of the surface displacements is the cross-spectral density, we obtain for a body with uniform loss

$$S_\varphi^{\mathrm{single}}(\omega)=\frac{16}{\pi^2}\frac{k^2}{w^4}\int\int dS\int\int dS'\, e^{-2|\vec{r}|^2/w^2} e^{-2|\vec{r}\,'|^2/w^2} \times\langle \delta z(\vec{r})\,\delta z(\vec{r}\,')\rangle_\omega. \tag{12}$$

Equation (12) is the desired result, relating the spectral density of phase noise induced on a Gaussian beam by a single mirror reflection at finite temperature to the elastic properties of the mirror. Before evaluating this expression for the specific Green's function obtained for a half-infinite mirror, we develop similar expressions for the cases of a Fabry-Perot interferometer in Sec. III and a multipass delay line in Sec. IV.

III. PHASE NOISE OF A GAUSSIAN BEAM EMERGING FROM A FABRY-PEROT INTERFEROMETER

Here, we generalize the noise formula (12) from the single-bounce case to the light reflected from a Fabry-Perot interferometer assuming the end mirror is a perfect reflector, i.e., has a unit power reflection coefficient $(R_E=1)$, and the input mirror has a power reflection coefficient R_I. The field reflection coefficient of the input mirror can the be written as $r_I=\sqrt{R_I}$, and the transmission in either direction through this mirror is it_I where $r_I^2+t_I^2=1$ for a lossless cavity. For an incident field E_0 the expression for the reflected field is, as usual [9],

$$\frac{E_{\rm ref}(t)}{E_0}=r_I-t_I^2\{e^{i\varphi_E(t-\tau)}+r_I e^{i[\varphi_E(t-\tau)+\varphi_I(t-2\tau)+\varphi_E(t-3\tau)]}+r_I^2 e^{i[\varphi_E(t-\tau)+\varphi_I(t-2\tau)+\varphi_E(t-3\tau)+\varphi_I(t-4\tau)+\varphi_E(t-5\tau)}+\cdots\}, \tag{13}$$

where $\varphi_E(t)$ and $\varphi_I(t)$ are the phase shifts due to the end and input mirrors, respectively, at time t, and τ is the single-pass transit time through the interferometer, i.e., $\tau=c/L$, where L is the distance between the Fabry-Perot mirrors. For small fluctuations, the phase shifts are small, and thus the exponentials in Eq. (13) can be approximated by the first-order expansion term. Arranging the resulting terms by time delay yields

$$E_{\rm ref}/E_0=r_I-\frac{t_I^2}{1-r_I}\left\{1+i\sum_{n=1,3,\ldots}[r_I^{(n-1)/2}\varphi_E(t-n\tau)]+i\sum_{n=2,4,\ldots}[r_I^{n/2}\varphi_I(t-n\tau)]\right\}. \tag{14}$$

Assuming that the sum over the phase shifts is still small, we can determine to first order the overall phase in Eq. (14), and find that

$$\varphi(t)=(1+r_I)[\Delta\varphi_E(t)+\Delta\varphi_I(t)], \tag{15}$$

where

$$\Delta\varphi_E(t)=\varphi_E(t-\tau)+r_I\varphi_E(t-3\tau)+\cdots=\sum_{n=1}^{\infty}r_I^{n-1}\varphi_E(t-(2n-1)\tau)$$
$$\Delta\varphi_I(t)=r_I\varphi_I(t-2\tau)+r_I^2\varphi_I(t-4\tau)+\cdots=\sum_{n=1}^{\infty}r_I^{n}\varphi_I(t-2n\tau). \tag{16}$$

With Eq. (15), the autocorrelation function of the phase noise is

$$\langle\varphi(t)\varphi(0)\rangle=(1+r_I)^2\{\langle\Delta\varphi_E(t)\Delta\varphi_E(0)\rangle+\langle\Delta\varphi_I(t)\Delta\varphi_I(0)\rangle\}, \tag{17}$$

where we use the statistical independence of the fluctuations in the two mirrors to eliminate the cross terms and assumed stationary statistics. By inserting Eqs. (16) into Eq. (17), moving the expectation operator inside the summations, and Fourier transforming the autocorrelation function, we obtain the power spectral density of the phase noise

$$S_\varphi^{\rm FP}(\omega)=\int_{-\infty}^{\infty}dt e^{i\omega t}\langle\varphi(t)\varphi(0)\rangle=(1+r_I)^2\left\{\sum_{n=1}^{\infty}\sum_{p=1}^{\infty}r_I^{n+p-2}e^{i\omega 2(n-p)\tau}\times\int_{-\infty}^{\infty}dt\,e^{i\omega t}\langle\varphi_E(t)\varphi_E(0)\rangle+\sum_{n=1}^{\infty}\sum_{p=1}^{\infty}r_I^{n+p}e^{i\omega 2(n-p)\tau}\times\int_{-\infty}^{\infty}dt\,e^{i\omega t}\langle\varphi_I(t)\varphi_I(0)\rangle\right\}, \tag{18}$$

where we have used the Fourier shift theorem. Noting that the Fourier transforms appearing in Eq. (18) are independent of the double sums; they can be factored out of the sum, as a direct consequence of the statistical stationarity. The integrals yield the single-reflection phase noises $S_\varphi^E(\omega)$ and $S_\varphi^I(\omega)$ for the end and input mirrors, respectively, according to the definition (11). Summing the remaining series, we find, after rationalizing the denominator and multiplying out the terms, that

$$S_\varphi^{\rm FP}(\omega)=\left[\frac{(1+r_I)^2}{(1+r_I^2)}\right]\left[1-\frac{2r_I}{1+r_I^2}\cos(2\omega\tau)\right]^{-1}\times[S_\varphi^E(\omega)+r_I^2S_\varphi^I(\omega)]. \tag{19}$$

Equation (19) is the first principal result of this paper, expressing the phase noise of the Gaussian beam emerging from a Fabry-Perot interferometer in terms of the input mirror reflection coefficient r_I, the transit time τ, and the single-reflection phase noises $S_\varphi^E(\omega)$ and $S_\varphi^I(\omega)$. Since $r_I\approx 1$, the denominator of Eq. (19) can be near zero when $\omega\tau=n\pi$, corresponding to the constructive accumulation of the phase noise at these frequencies, for which the surface fluctuation δz modulates the laser beam in resonance with the Fabry-Perot response.

IV. PHASE NOISE OF A GAUSSIAN BEAM PASSING THROUGH A DELAY LINE

The phase fluctuations for a beam passing through a delay line in which the light bounces between two mirrors, sampling a series of different points on each mirror at a series of different times, can be obtained by summing up a series of phase contributions for a single bounce as given in Eq. (8), with the time and space coordinates chosen appropriately for each bounce. More precisely, for a delay line in which the beam bounces N times at positions $\vec{r}_{n'}$ on the surface S' of the end mirror, and $N-1$ times at positions $\vec{\rho}_{p'}$ on the surface σ' of the input mirror, with transit time τ between the mirrors, the total phase noise can be written

$$\varphi(t)=\varphi_E(t)+\varphi_I(t)=\sum_{n'=1}^{N}\varphi_E^{\text{single}}(\vec{r}_{n'},t_{n'})+\sum_{p'=1}^{N-1}\varphi_I^{\text{single}}(\vec{\rho}_{p'},t_{p'}),\tag{20}$$

where the second forms are obtained from the first by the use of the single-mirror phase function given in Eq. (8), except that it is generalized to a Gaussian beam centered on a point $\vec{r}_n$ at a time t_n,

$$\varphi^{\text{single}}(\vec{r}_n,t_n)=\frac{4k}{\pi w^2}\int\int dS'\,e^{-2|\vec{r}'-\vec{r}_n|^2/w^2}\delta z(\vec{r}',t_n),\tag{21}$$

and the definitions of the bounce times $t_{n'}=t-(2n'-1)\tau$ and $t_{p'}=t-2p'\tau$.

The autocorrelation function for the phase is

$$\begin{aligned}\langle\varphi(t)\varphi(t')\rangle&=\langle\varphi_E(t)\varphi_E(t')\rangle+\langle\varphi_I(t)\varphi_I(t')\rangle\\&=\left\langle\sum_{n'=1}^{N}\varphi_E^{\text{single}}(\vec{r}_{n'},t_{n'})\sum_{n''=1}^{N}\varphi_E^{\text{single}}(\vec{r}_{n''},t'_{n''})\right\rangle\\&\quad+\left\langle\sum_{p'=1}^{N-1}\varphi_I^{\text{single}}(\vec{\rho}_{p'},t_{p'})\sum_{p'=1}^{N-1}\varphi_I^{\text{single}}(\vec{\rho}_{p''},t'_{p''})\right\rangle\\&=\sum_{n'=1}^{N}\sum_{n''=1}^{N}\langle\varphi_E^{\text{single}}(\vec{r}_{n'},t_{n'})\varphi_E^{\text{single}}(\vec{r}_{n''},t'_{n''})\rangle\\&\quad+\sum_{p'=1}^{N-1}\sum_{p''=1}^{N-1}\langle\varphi_I^{\text{single}}(\vec{\rho}_{p'},t_{p'})\varphi_I^{\text{single}}(\vec{\rho}_{p''},t'_{p''})\rangle,\end{aligned}\tag{22}$$

where the first form of Eq. (22) takes advantage of the independence of the fluctuations on the two mirrors, and the second uses the second form of Eq. (20). Fourier transforming the autocorrelation function, Eq. (22), with respect to $(t-t'\equiv T)$ yields the power spectral density of the phase noise. To evaluate this quantity, consider a typical term,

$$\begin{aligned}O^E_{n'n''}&\equiv\int_{-\infty}^{\infty}dT e^{i\omega T}\langle\varphi_E^{\text{single}}(\vec{r}_{n'},t_{n'})\varphi_E^{\text{single}}(\vec{r}_{n''},t'_{n''})\rangle\\&=\left(\frac{4k}{\pi w^2}\right)^2\int\int dS'\int\int dS''\\&\quad\times e^{-2|\vec{r}'-\vec{r}_{n'}|^2/w^2}e^{-2|\vec{r}''-\vec{r}_{n''}|^2/w^2}\\&\quad\times\int_{-\infty}^{\infty}dT\,e^{i\omega T}\langle\delta z_E(\vec{r}',t_{n'})\delta z_E(\vec{r}'',t'_{n''})\rangle,\end{aligned}\tag{23}$$

where in the second form we have used the single-mirror phase function as in Eq. (12) but with the necessary modification Eq. (21). Noting that $t_{n'}-t'_{n''}=t-t'+2(n''-n')\tau$ and applying the Fourier shift theorem, we see that

$$\begin{aligned}O^E_{n'n''}&=\left(\frac{4k}{\pi w^2}\right)^2e^{i2(n'-n'')\tau\omega}\int\int dS'\int\int dS''\\&\quad\times e^{-2(|\vec{r}'-\vec{r}_{n'}|^2+|\vec{r}''-\vec{r}_{n''}|^2)/w^2}\langle\delta z_E(\vec{r}')\delta z_E(\vec{r}'')\rangle_\omega\\&=e^{i2(n'-n'')\tau\omega}S^E_\varphi(\omega,\vec{r}_{n'},\vec{r}_{n''}),\end{aligned}\tag{24}$$

where $S_\varphi(\omega,\vec{r}_1,\vec{r}_2)$ is the cross-spectral density of the phase fluctuations of Gaussian beams centered at $\vec{r}_1$ and $\vec{r}_2$,

$$\begin{aligned}S_\varphi(\omega,\vec{r}_1,\vec{r}_2)&\equiv\frac{16}{\pi^2}\frac{k^2}{w^4}\int\int dS'\int\int dS''\\&\quad\times e^{-2|\vec{r}'-\vec{r}_1|^2/w^2}e^{-2|\vec{r}''-\vec{r}_2|^2/w^2}\langle\delta z(\vec{r}')\delta z(\vec{r}'')\rangle_\omega,\end{aligned}\tag{25}$$

which is an analog of Eq. (12) but generalized to two discrete laser-beam spots, $\vec{r}_1$ and $\vec{r}_2$ in accordance with Eq. (21). It is symmetrical under the interchange $\vec{r}_1\leftrightarrow\vec{r}_2$ due to the Onsager theorem [10]. With these quantities the phase noise spectral density for the end mirror is given by

$$\begin{aligned}S^E_\varphi(\omega)&=\sum_{n'=1}^{N}\sum_{n''=1}^{N}O^E_{n'n''}\\&=\sum_{n'=1}^{N}\sum_{n''=1}^{N}e^{i2(n'-n'')\tau\omega}S^E_\varphi(\omega,\vec{r}_{n'},\vec{r}_{n''})\\&=\sum_{n=1}^{N}S^E_\varphi(\omega,\vec{r}_n,\vec{r}_n)+2\sum_{n=2}^{N}\sum_{q=1}^{n-1}\cos[2(n-q)\tau\omega]\\&\quad\times S^E_\varphi(\omega,\vec{r}_n,\vec{r}_q),\end{aligned}\tag{26}$$

where the last line explicitly separates the equal-point terms for $n'=n''$ and uses the aforementioned Onsager relation to simplify and reindex the double sum. Following the same procedure for the input mirror and combining with Eq. (26) for the end mirror, we arrive at an expression for the spectral density of phase noise imposed on a Gaussian beam passing through the delay line,

$$\begin{aligned}S_\varphi(\omega)&=S^E_\varphi(\omega)+S^I_\varphi(\omega)\\&=\sum_{n=1}^{N}S^E_\varphi(\omega,\vec{r}_n,\vec{r}_n)+2\sum_{n=2}^{N}\sum_{q=1}^{n-1}\cos[2(n-q)\tau\omega]\\&\quad\times S^E_\varphi(\omega,\vec{r}_n,\vec{r}_q)+\sum_{n=1}^{N-1}S^I_\varphi(\omega,\vec{\rho}_n,\vec{\rho}_n)\\&\quad+2\sum_{n=2}^{N-1}\sum_{q=1}^{n-1}\cos[2(n-q)\tau\omega]S^I_\varphi(\omega,\vec{\rho}_n,\vec{\rho}_q).\end{aligned}\tag{27}$$

Equation (27) is the second principal result of this paper.

V. HALF-INFINITE MIRRORS WITH UNIFORM LOSS

A useful model system simpler to treat than mirrors with realistic aspect ratios, but retaining much of the essential physics, is a semi-infinite mirror [5]. In the next three subsections, we evaluate the cross-spectral density of the displacement noise for a single bounce from a mirror, for reflection from a Fabry-Perot interferometer, and for passage through a delay line, Eqs. (12), (19), and (27), respectively. For each of these expressions we need the cross-spectral density of the displacement noise of the mirrors, as expressed by the fluctuation dissipation theorem in Eq. (3) or Eq. (4). The essential input for these three calculations is the static elastic Green's function for an isotropic half-space [10],

$$\chi_{zz}^{\mathrm{static}}(\vec{r}\,'-\vec{r}\,'')=\frac{1-\sigma^2}{\pi E}\,\frac{1}{|\vec{r}\,'-\vec{r}\,''|}. \tag{28}$$

When this expression is substituted into Eq. (4), we obtain for the cross-spectral density of the surface displacements

$$\langle \delta z(\vec{r}\,')\,\delta z(\vec{r}\,'')\rangle_\omega=\frac{2k_BT}{\omega}\,\phi(\omega)\,\frac{1-\sigma^2}{\pi E}\,\frac{1}{|\vec{r}\,'-\vec{r}\,''|}, \tag{29}$$

where σ is the Poisson ratio, E the Young's modulus, and $\vec{r}\,'=(x',y')$ is the position on the mirror surface.

A. Phase noise after a single bounce from a half-infinite mirror

Substituting Eq. (29) for the cross-spectral density of the displacement noise into Eq. (12) for the phase noise after a single bounce from a mirror, we find

$$S_\varphi^{\mathrm{single}}(\omega)=\left(\frac{32k_BT}{\pi^3}\,\frac{\phi(\omega)}{\omega}\,\frac{k^2}{w^4}\right)\left[\frac{(1-\sigma^2)}{E}\int\int dS' \times\int\int dS'' e^{-2|\vec{r}\,'|^2/w^2}e^{-2|\vec{r}\,''|^2/w^2}\frac{1}{|\vec{r}\,'-\vec{r}\,''|}\right]. \tag{30}$$

The double surface integral can be performed explicitly as shown in the Appendix, yielding the result

$$S_\varphi^{\mathrm{single}}(f)=\frac{4k_BT}{\pi^{3/2}}\,\frac{\phi(f)}{f}\,\frac{k^2}{w}\,\frac{(1-\sigma^2)}{E}. \tag{31}$$

This single-bounce result (31) agrees with Eq. (15) of Levin [5] when corrected for an algebraic error found by Liu and Thorne [8] as shown in their Eq. (59). Note also that Levin's result is a single-sided spectral density for the displacement noise and our result is a double-sided spectral density. Moreover, our result is for the phase noise, which must be divided by $4k^2$ to produce the result for the displacement noise.

B. Phase noise after reflection from a Fabry-Perot interferometer

Recall that Eq. (19) for the phase noise after reflection from a Fabry-Perot interferometer requires the single-bounce phase-noise spectral densities S_φ^E and S_φ^I, each given by Eq. (31),

$$S_\varphi^{\mathrm{FP}}(\omega)=\left[\frac{(1+r_I)^2}{(1+r_I^2)}\right]\left[1-\frac{2r_I}{1+r_I^2}\cos(2\omega\tau)\right]^{-1} \times[S_\varphi^E(\omega)+r_I^2S_\varphi^I(\omega)]. \tag{32}$$

This expression shows that the Brownian motions of the two mirrors impose on the light-beam phase noise, which is filtered by the Fabry-Perot response function.

C. Phase noise after passing through a delay line

The phase-noise spectral density for a delay line received considerable attention from the gravitational wave interferometry group at the Max-Planck-Institute for Quantumoptics in Garching, Germany in the 1980s, where the potential advantage of delay lines was recognized from experimental results on their 30-m interferometer, and was discussed qualitatively in terms of a modal picture [11]. This work resulted in a series of unpublished reports and is discussed briefly in Ref. [12] by the same group.

The complexity of the analysis for the delay line is considerably greater than that for a Fabry-Perot interferometer. The point of departure is the evaluation of expression (27) for the cross-spectral density of displacement noise in the mirror, which requires the two-point phase-noise correlation function (25) in place of Eq. (30), i.e.,

$$S_\varphi(\omega,\vec{r}_1,\vec{r}_2)=\frac{32k_BT}{\pi^3}\,\frac{\phi(\omega)}{\omega}\,\frac{k^2}{w^4}\,\frac{(1-\sigma^2)}{E}\int\int dS' \times\int\int dS'' e^{-2|\vec{r}\,'-\vec{r}_1|^2/w^2}e^{-2|\vec{r}\,''-\vec{r}_2|^2/w^2} \times\frac{1}{|\vec{r}\,'-\vec{r}\,''|}. \tag{33}$$

Fortunately, these surface integrations can be performed explicitly, as shown in the Appendix, yielding

$$S_\varphi(f,\vec{r}_1,\vec{r}_2)=\frac{4k_BT}{\pi^{3/2}}\,\frac{\phi(f)}{f}\,\frac{k^2}{w}\,\frac{(1-\sigma^2)}{E}\,e^{-|\vec{r}_1-\vec{r}_2|^2/2w^2} \times I_0(|\vec{r}_1-\vec{r}_2|^2/2w^2), \tag{34}$$

where I_0 is the modified Bessel function of the first kind. Substituting this expression into Eq. (27), we obtain the final result for the phase noise after passing through a delay line,

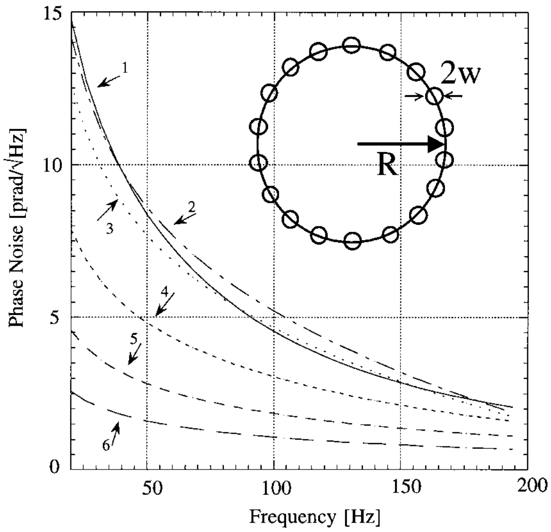

FIG. 1. Comparison of the phase noise from a delay line and a Fabry-Perot interferometer. The solid curve (1) is for a 4-km Fabry-Perot interferometer with an input mirror power reflectivity of $R_I = 0.97$ and an end mirror power reflectivity of $R_E = 1.00$. Curves 2, 3, 4, 5, and 6 correspond to 4-km delay lines all with 130 spots on the end mirror and laser-beam spots of $1/e$ field radius w in a pattern with their centers on a circles of radius $R = w/3$ (2), $R = 2w/3$ (3), $R = 5w/2$ (4), $R = 10w$ (5), and $R = 20w$ (6), where $w = 3.5$ cm, the spot size used for both mirrors of the Fabry-Perot interferometer. The mirror Q is assumed to be 3×10^8 and the material properties are those of sapphire $E = 71.8$ GP and $\sigma = 0.16$. However, we are treating sapphire as isotropic for the purpose of this illustration and assuming a single loss function.

$$S_\phi(f) = S_\phi^{\text{single}}(f)\Bigg\{(2N-1) + 2\sum_{n=2}^{N}\sum_{q=1}^{n-1}\cos[2(n-q)\tau\omega] \times e^{-|\vec{r}_n-\vec{r}_q|^2/2w^2} I_0(|\vec{r}_n-\vec{r}_q|^2/2w^2) + 2\sum_{n=2}^{N-1}\sum_{q=1}^{n-1}\cos[2(n-q)\tau\omega] e^{-|\vec{\rho}_n-\vec{\rho}_q|^2/2w^2} \times I_0(|\vec{\rho}_n-\vec{\rho}_q|^2/2w^2)\Bigg\}, \quad (35)$$

where we have used Eq. (31) to express the prefactor.

D. Comparison of a Fabry-Perot interferometer and a delay line

Figure 1 compares the phase noise from a Fabry-Perot interferometer to that of several delay lines using the same storage time for the Fabry-Perot interferometer as that proposed for LIGO II [13]. The number of bounces used to compute the delay-line noise is the number of bounces that will produce the same phase variation of the light for a sinusoidal motion of the end mirror at frequencies well below the bandwidth of the Fabry-Perot interferometer. The laser beam spots are all of $1/e$ field radius w and are positioned on the delay-line mirrors with their centers evenly spaced in angle and lying on a circle of radius R as shown in Fig. 1. As the spot circle radius becomes comparable to the beam spot size on the mirrors, the phase noise from the delay line approaches that of the Fabry-Perot interferometer. When the spots are largely overlapping, the phase noise for the delay line is still less than that of the Fabry-Perot interferometer between 40 and 175 hertz, after which the delay line is noisier. When the spots are not overlapping appreciably the thermal noise of the delay line is less than that of the Fabry-Perot interferometer. However, in these calculations the spatial correlation scale is the size of the light beams on the mirror. Any increase in the spatial correlation due to the finite size of the mirrors is ignored in this calculation for a half-infinite mirror. Thus it seems likely that the noise in a delay line made from mirrors with a more realistic aspect ratio would be larger than is calculated here. Evaluation of the thermal noise in a model system of finite transverse dimension is currently being undertaken to clarify this point.

VI. CONCLUSION

The primary results of this paper are the relations (19) and (27). They relate laser-beam phase noises to stochastic mirror surface vibrations, generalizing the known single-reflection result (12) to a Fabry-Perot interferometer (19) and to an optical delay line (27), respectively. Specifically, the relations facilitate evaluations of the phase-noise fluctuation correlation of a Gaussian laser beam in terms of a two-point cross-spectral correlation function of the mirror surface displacements. In particular, when the surface fluctuation is dominated by the thermal excitation, one can calculate intrinsic thermal noises of the laser-beam phase, with the help of the fluctuation-dissipation relation [Eq. (3) or (4)], for Fabry-Perot interferometers and optical delay lines.

Section V presents example calculations of thermal noise for isotropic semi-infinite mirrors. The resulting expressions are given in Eq. (31) for the single-reflection case, in Eq. (32) for a Fabry-Perot interferometer, and in Eq. (35) for a delay line. Our results for a single bounce from a mirror are basically the same results obtained by Levin [5], Bondu, Hello, and Vinet [7], and Liu and Thorne [8]. Moreover, the results obtained for the Fabry-Perot resonators are consistent with intuitive expectations. The comparison between the Fabry-Perot interferometer and the delay line indicates that there may exist parameter regions where delay lines could be quieter than Fabry-Perot interferometers, depending on the relative sizes of the spot circle radius, the beam spot, and the mirror. However, these results must be used with caution since the fact that the laser beam spots are all near the edge of the mirror, rather than at the center, could lead to a higher degree of spatial correlation in actual mirrors than the case for the half-infinite mirrors. While there may be thermal noise issues that a semi-infinite mirror model fails to address, it remains a useful model system and will be applied in future work to several problems, including nonuniform loss and elastic anisotropy as would be encountered with crystalline materials that may be chosen for the mirrors in advanced receivers. By an appropriate generalization of the Green's function, these would be amenable to an analysis similar to that described here.

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APPENDIX

In evaluating Eqs. (30) and (33), we have used the relations

$$\int\int dS'\int\int dS'' e^{-2|\vec{r}'|^2/w^2}e^{-2|\vec{r}''|^2/w^2}\frac{1}{|\vec{r}'-\vec{r}''|}=\frac{\pi^{5/2}}{4}w^3 \tag{A1}$$

and

$$\int\int dS'\int\int dS'' e^{-2|(\vec{r}'-\vec{r}_p)/w|^2}e^{-2|(\vec{r}''-\vec{r}_q)/w|^2}\frac{1}{|\vec{r}'-\vec{r}''|} = \frac{\pi^{5/2}}{4}w^3 e^{-(\vec{r}_p-\vec{r}_q)^2/2w^2}I_0((\vec{r}_p-\vec{r}_q)^2/2w^2). \tag{A2}$$

Below, we will derive Eq. (A2), while Eq. (A1) follows Eq. (A2) as a special case when $\vec{r}_p=\vec{r}_q=\vec{0}$.

The derivation starts with the integral representation

$$\frac{1}{|\vec{r}'-\vec{r}''|}=\frac{1}{w}\sqrt{\frac{2}{\pi}}\int_0^\infty d\lambda\,\lambda^{-1/2}e^{-2\lambda(\vec{r}'-\vec{r}'')^2/w^2}. \tag{A3}$$

Using Eq. (A3), shifting variables as $\vec{r}'\to\vec{r}'+\vec{r}_p$ and $\vec{r}''\to\vec{r}''+\vec{r}_q$, and interchanging the integration order, we turn the left-hand side of Eq. (A2) into

$$\frac{1}{w}\sqrt{\frac{2}{\pi}}\int_0^\infty d\lambda\,\lambda^{-1/2}\int\int dS' \times\int\int dS'' e^{-2\{\vec{r}'^2+\vec{r}''^2+\lambda(\vec{r}'-\vec{r}''+\vec{r}_{pq})^2\}/w^2}, \tag{A4}$$

where $\vec{r}_{pq}\equiv\vec{r}_p-\vec{r}_q$. Then, by regrouping the exponent as

$$\vec{r}'^2+\vec{r}''^2+\lambda(\vec{r}'-\vec{r}''+\vec{r}_{pq})^2 =(1+\lambda)\left\{\vec{r}'-\frac{\lambda}{1+\lambda}(\vec{r}''-\vec{r}_{pq})\right\}^2+\frac{1+2\lambda}{1+\lambda} \times\left(\vec{r}''-\frac{\lambda}{1+2\lambda}\vec{r}_{pq}\right)^2+\frac{\lambda}{1+2\lambda}\vec{r}_{pq}^2, \tag{A5}$$

we perform the surface integrations as Gaussian integrations, finding that

$$[\text{Eq. }(A4)]=\frac{1}{w}\sqrt{\frac{2}{\pi}}\left(\frac{\pi w^2}{2}\right)^2\int_0^\infty d\lambda \times\lambda^{-1/2}\frac{1}{1+2\lambda}e^{-[2\lambda/(1+2\lambda)](\vec{r}_{pq}^2/w^2)}. \tag{A6}$$

A change of the integration variable from λ to θ defined by $2\lambda/(1+2\lambda)=(1+\cos\theta)/2$ turns the parameter integral of Eq. (A6) into the canonical integral representation of the modified Bessel function I_0 of the first kind, namely,

$$\int_0^\infty d\lambda\frac{\lambda^{-1/2}}{1+2\lambda}e^{-[2\lambda/(1+2\lambda)](\vec{r}_{pq}^2/w^2)} =\frac{1}{\sqrt{2}}e^{-\vec{r}_{pq}^2/2w^2}\int_0^\pi d\theta\, e^{-(\vec{r}_{pq}^2/2w^2)\cos\theta} =\frac{\pi}{\sqrt{2}}e^{-\vec{r}_{pq}^2/2w^2}I_0(\vec{r}_{pq}^2/2w^2). \tag{A7}$$

Equation (A2) now follows Eqs. (A6) and (A7).

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